Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 20, 2012

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Crazy dice)**.** An eccentric friend of yours has a pair of fair, six-sided dice, all of whose faces are labeled with positive integers. When the two dice are rolled, the sum of the top faces has the same probability distribution as that of a standard pair of six-sided dice. In other words, the possible outcomes range from 2 to 12, and each of these occurs with the same probability as for a standard pair of dice.

However, your friend's dice are **not** a standard pair; in fact, the faces of the first die are labeled $1, 2, 2, 3, 3, 4$. What are the labels on the faces of the other die? **Write them down in increasing order.**

Problem 2 (Squares)**.** It is known that there exists a unique positive integer $n > 1$ such that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = m^2$$

is a square of another integer m. **Find n.**

Problem 3 (Exciting tournaments)**.** Sixteen ($16 = 2^4$) teams participate in a single-elimination tournament of four rounds. That is, each team plays some other team in the first round, and the winners advance to the second round, and so forth. Suppose the teams are currently ranked best to worst from #1 to #16 and the higher-ranked team always wins in every game. The tournament designers, instead of using the rankings, chose the team matchups completely randomly. What is the probability that teams #1–#8 all advance to the quarterfinals (second round), teams #1–#4 all advance to the semifinals (third round), and teams #1 and #2 meet in the final (fourth round)?

Express your answer as a fraction in simplified form. (The graders will not do arithmetics for you.)

Authors. Problem 1 was written by Paul Pollack, problem 2 by Valery Alexeev, and problem 3 by Boris Alexeev.