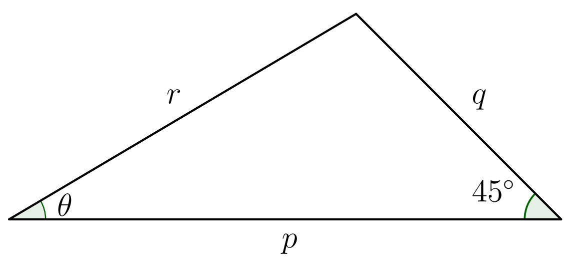
1785

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 21, 2017

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Double trouble)**.** Recall that the functions $c(\theta)$ and $s(\theta)$ were defined, for $0^\circ < \theta < 135^\circ$, as follows: Given θ, construct a triangle with angles θ° and 45°, and side lengths as indicated in this diagram:



Then $c(\theta) = \dfrac{p}{r}$ and $s(\theta) = \dfrac{q}{r}$.

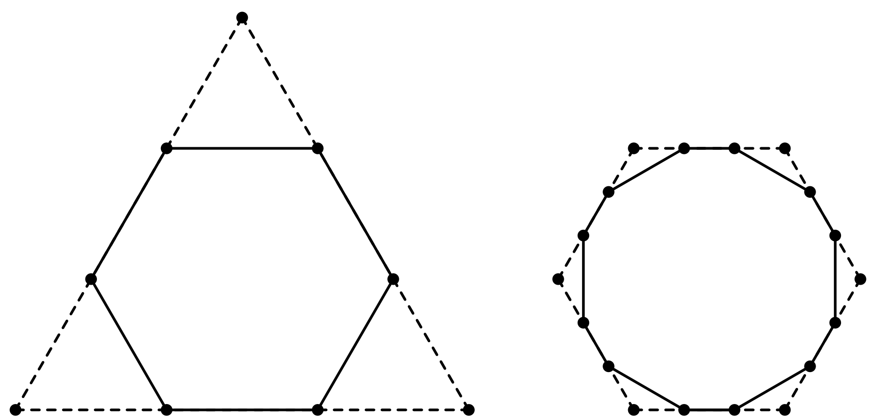
Find the double angle formula for $s(\theta)$, expressed as a polynomial in $c(\theta)$. For instance, a reasonable — but incorrect! — answer might look like $s(2\theta) = c(\theta)^3 + 2c(\theta)^2 - c(\theta) - 2$.

Problem 2 (Strike that. Reverse it.)**.** There are infinitely many pairs of distinct positive rational numbers x, y satisfying

$$x^y = y^x.$$

One example, with integers x and y, is $x = 2, y = 4$. Suppose that x, y is such a pair with neither x nor y an integer. Write $x = a/b$ and $y = c/d$, with a/b and c/d in lowest terms. (Recall that m/n is in *lowest terms* if $n > 0$ and the greatest common divisor of m and n is 1.) What is the smallest possible value of $b + d$?

Problem 3 (Cutting corners)**.** Begin with an equilateral triangle. Trisect each of its sides and cut off the corners. Take the resulting figure, and again trisect each of its sides and cut off the corners. If you repeat this process infinitely many times, what is the ratio of the area of the resulting figure to the area of the original triangle? The first two iterations are pictured below.



RETURN THIS SHEET

Team ID:

Team name:

Answer 1:

Answer 2:

Answer 3: