University of Georgia High School Math Tournament

2005

Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

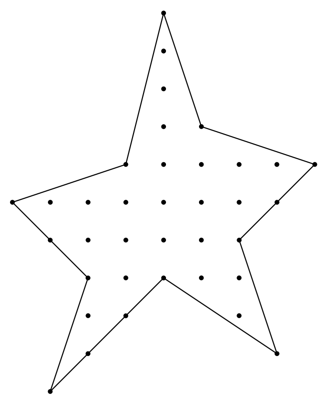
5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit through the rear doors.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. The star below is drawn on the standard rectangular grid on which every square has size 1 by 1. What is the area of the star?



(A) 26.5 (B) 27 (C) 27.5 (D) 28 (E) None of the above.

Problem 2. The Fibonacci numbers

$$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$$

have the property that every number of this sequence, starting with the third, is the sum of two previous numbers (for example, $8 = 5 + 3$). What is the greatest common divisor of the 2004th and 2005th Fibonacci numbers?

(A) 1 (B) 2 (C) 3 (D) 5 (E) None of the above

Problem 3. As has been recently discovered, there are three types of amoebas on Mars: types A, B and C. It is also known that if two amoebas of different types fuse together, they form an amoeba of the third type (for example, a type B amoeba and a type C amoeba fuse to form a type A amoeba). After a certain time, there remains only one amoeba. If originally there were 20 type A, 21 type B and 22 type C amoebas, what type is the last remaining amoeba?

(A) A (B) B (C) C (D) The answer is not uniquely determined. (E) More than one amoeba must remain at the end (for example, if two type A amoebas remain, no more fusion can occur) if one begins with the specified numbers.

Problem 4. $x + \dfrac{1}{x} = 3$. What is $x^4 + \dfrac{1}{x^4}$?

(A) 7 (B) 9 (C) 47 (D) 81 (E) None of the above

Problem 5. Let $f(n) = n^3$ and define the function $g(n)$ by the formula

$$g(n) = f(n+1) - f(n)$$

What is the average of the 10 numbers $g(0), g(1), \ldots, g(9)$?

(A) 64 (B) 81 (C) 95 (D) 105 (E) None of the above

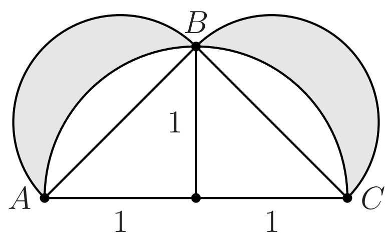
Problem 6. Let $P(x)$ be a polynomial such that

$$\big(P(x)\big)^2 = 1 - 2\,x + 5\,x^2 - 4\,x^3 + 4\,x^4$$

What is $|P(1)|$?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

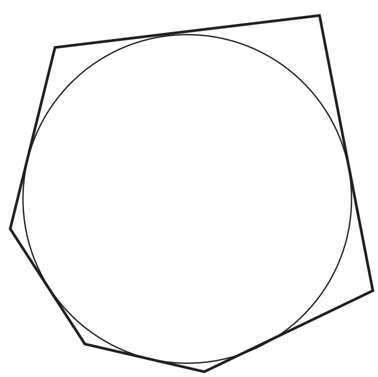
Problem 7. On the picture below, the diameters of the three semicircles are sides of the right triangle ABC. The angles A and C are 45 degrees each.



Find the shaded area.

(A) 1 (B) $\frac{\pi}{2} - 1$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{2}$ (E) $\frac{3\pi}{2} - 1$

Problem 8. A circumscribed hexagon has sides 2, 3, 5, 7, 9 and x in clockwise order. What is x?



(A) 2 (B) 4 (C) 6 (D) 8 (E) None of the above

Problem 9. What is the last digit of

$$7^{7^{7^7}}\ ?$$

(A) 1 (B) 3 (C) 5 (D) 7 (E) 9

Problem 10. There are 8 empty chairs in a row. In how many ways can one seat 3 people so that no two of them sit next to one another?

(A) 20 (B) 24 (C) 56 (D) 720 (E) None of the above

Problem 11. What is $(1+i)^{11}$?

(A) $16+16i$ (B) $-16+16i$ (C) $32+32i$ (D) $-32+32i$ (E) None of the above

Problem 12. In a mythical country people have only two kinds of coins: 7 and 9 cents. What is the largest amount that *cannot* be made using these coins?

(A) 38 (B) 40 (C) 47 (D) 48 (E) None of the above

Problem 13. The Fibonacci numbers $f_1, f_2, \ldots$ are $1, 1, 2, 3, 5, 8, 13, \ldots$

(every number is the sum of the previous two). Find

$$\sum_{n=1}^{\infty} \frac{f_n}{2^n}$$

(A) 1 (B) 2 (C) 3 (D) 4 (E) None of the above

Problem 14. How many times during a day (24 hours) do the hour hand and the minute hand on the clock point in opposite directions?

(A) 2 (B) 22 (C) 24 (D) 20 (E) None of the above

Problem 15. Take a random irrational number such as $\pi = 3.1415926535\cdots$, and add up the first k digits after the decimal point. For example, with $k = 5$, $1+4+1+5+9 = 20$. What is the probability that, for some k, this sum is 2005? The answer is closest to:

(A) 0 (B) 1/10 (C) 1/9 (D) 1/5 (E) 2/9

Problem 16. Which of the following numbers can not be written as a difference of two perfect squares?

(A) 20,000,002 (B) 20,000,003 (C) 20,000,004 (D) None of these (E) More than one of these

Problem 17. Compute

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + 7 \cdot 7!$$

(A) 40,019 (B) 40,119 (C) 40,329 (D) 3,528,799 (E) None of the above

Problem 18. How many numbers between 1 and 2004 are relatively prime to 2005 (i.e., the two numbers have greatest common divisor 1)?

(A) 800 (B) 805 (C) 1604 (D) 2000 (E) None of the above

Problem 19. When you divide the polynomial $x^{2005} + x + 2$ by $x^2 - 1$, what will be the remainder?

(A) $2x - 2$ (B) $-2x - 2$ (C) $-2x - 2$ (D) 2 (E) None of the above

Problem 20. Two ships move with constant speeds and directions. At noon, 2 p.m., and 3 p.m. respectively the distance between them was respectively 5, 7, and 2 miles. What was the distance between them at 1 p.m.?

(A) 6 (B) 10 (C) $\sqrt{50}$ (D) $\sqrt{56}$ (E) None of the above

Problem 21. Five points are placed on a sphere of radius 1 such that the sum of the squares of the pairwise distances between them are maximized. (There are 10 terms in this sum.) What is this sum?

(A) 20 (B) 21 (C) 24 (D) 30 (E) None of the above

Problem 22. What is the 100th smallest positive integer that can be written as the sum of **distinct** powers of 3, i.e., $1, 3, 9, 27, \ldots$?

(A) 969 (B) 973 (C) 974 (D) 981 (E) None of the above

Problem 23. Find

$$\cos 20^\circ \cos 40^\circ \cos 60^\circ \cos 80^\circ.$$

(A) $1/16$ (B) $\sqrt{2}/24$ (C) $1/12$ (D) $1/8$ (E) None of the above

Problem 24. Ten people are standing in line to buy movie tickets that cost \$5. Five of them have only \$5 bills and the other five have only \$10 bills. The cashier has absolutely no change, so a person with a \$5 bill must come before a person with a \$10 bill or else the sale will not happen.

What is the probability that they will happen to stand in line just right so that all ten will be able to buy tickets?

(A) $1/10$ (B) $1/6$ (C) $1/5$ (D) $1/2$ (E) None of the above

Problem 25. Let $\phi(n)$ denote *Euler's function*, the number of integers $1 \leq i < n$ that are relatively prime to n. (For example, $\phi(9) = 6$ and $\phi(10) = 4$.) What is the last digit of the sum

$$\sum_{d|2005} \phi(d)$$

which goes over all (positive integral) divisors d of 2005 including 1 and 2005.

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D) 6 or 7 (E) 8 or 9

Authors. Written by Valery Alexeev and Boris Alexeev ©2005, with assistance by Ted Shifrin and Mo Hendon. Some problems were taken from N.B. Alfutova, A.B. Ustinov "Algebra and number theory for mathematical schools" published by Moscow Center for Continuing Mathematical Education, 2002.