Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 17, 2009

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

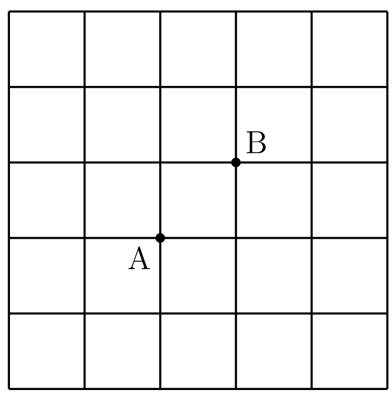
7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. John has a bunch of shoes in his closet, but only 2/3 of the left shoes have matching right shoes, and only 3/5 of the right shoes have matching left shoes. What fraction of the shoes are parts of matching pairs? (No shoe is part of two pairs.)

(A) 12/19 (B) 19/30 (C) 21/30 (D) 12/15 (E) None of the above

Problem 2. A curious tourist wants to go for a walk on the streets of the Old Town from his hotel (the point A on the map below) to the train station (the point B) using the longest way possible but never passing through the same point twice. (He can only move on the grid.)



If we consider any interval of length 1 to be a street, how many streets can the tourist traverse?

(A) 33 (B) 34 (C) 35 (D) 36 (E) None of the above

Problem 3. Every morning Joe walks for 1 mile along the tram tracks and counts the trams passing him from behind and coming towards him. During the year, he counted 100 of the former and 300 of the latter. Joe's speed is 3 mph. What is the tram's speed, in mph?

(A) 5 (B) 7 (C) 9 (D) 10 (E) None of the above

Problem 4. If a child is born in 2009, what will be the next year that both her age and the year are perfect squares?

(A) In 4 or 9 years (B) In 16 or 25 years (C) In 36 or 49 years (D) A different answer (E) Never

Problem 5. Consider the number

$$15! = 1 \cdot 2 \cdot 3 \cdots 14 \cdot 15$$

Add its digits to obtain a new number. Add its digits to obtain a new number, and continue this process until you get a single digit. What is it?

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D) 6 or 7 (E) 8 or 9

Problem 6. Two positive real numbers have an average of 10. Which of the following must be true about μ, the average of their reciprocals?

(A) $\mu = 10$ (B) $\mu = 1/10$ (C) μ can be any real number (D) $\mu \geq \frac{1}{10}$
(E) $\mu \leq \frac{1}{10}$

Problem 7. There is a peculiar species of worm which can either climb up 7 feet at once, or climb down 5 feet at once, but no more or less at any given step. What is the shortest pole that he can both climb up and climb down?

(A) 10 (B) 11 (C) 12 (D) 13 (E) more

Problem 8. Let n be the greatest number that is the product of some positive integers (possibly not distinct), such that the sum of these integers is 2009. Find the last digit of n.

(A) 0 or 1 (B) 2 or 3 (C) 4 or 5 (D) 6 or 7 (E) 8 or 9

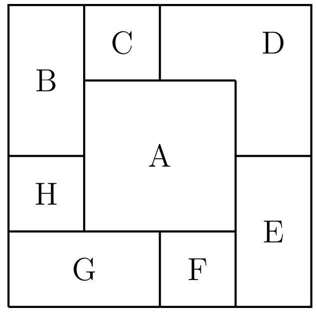
Problem 9. A group of hikers went on a 3.5-hour hike. In any consecutive one-hour period during their hike, they covered exactly two miles. What is the most distance they could have covered (in miles)?

(A) 6.5 (B) 7 (C) 7.5 (D) 8 (E) None of the above

Problem 10. If a star is born in 2009, how many times will it happen that both its age and the year are perfect squares?

(A) 1 (B) 2 (C) 3 (D) 4 (E) more

Problem 11. Eight square tissues of the same size were placed on the table, one by one, to form the picture shown below. In the order of placement, which was the tissue marked B?



(A) the second (B) the third (C) the fourth (D) the fifth (E) the sixth

Problem 12. How many two-digit numbers A have the property that the square of the sum of the digits of A equals the sum of the digits of A^2?

(A) 6 (B) 7 (C) 8 (D) 9 (E) None of the above

Problem 13. What is the slope of the line that bisects the angle in the first quadrant formed by the lines $y = 0$ and $y = 2x$?

(A) $\frac{\sqrt{5}+1}{2}$ (B) 1 (C) $\frac{\sqrt{5}-1}{2}$ (D) $\frac{1}{2}$ (E) $\frac{1}{3}$

Problem 14. How many positive integers n less than 2009 have the property that mn is **not** divisible by 2009 for **every** positive integer $m < 2009$?

(A) 1674 (B) 1680 (C) 1722 (D) 1960 (E) None of the above

Problem 15. What is the distance between a vertex and the center of a regular tetrahedron of side one?

(A) $\frac{2\sqrt{6}}{9}$ (B) $\frac{\sqrt{6}}{4}$ (C) $\frac{3\sqrt{3}}{8}$ (D) $\frac{\sqrt{6}}{3}$ (E) None of the above

Problem 16. What is the sum of the distances from the point $(1, 1)$ to the sides of the equilateral triangle with vertices at $(0, 0)$, $(4, 0)$, and $(2, 2\sqrt{3})$?

(A) 2 (B) $3(2\sqrt{3} - 1)$ (C) $2\sqrt{3}$ (D) 4 (E) $4\sqrt{3}$

Problem 17. If the side lengths of a right triangle are all integers and one is 2009, find their largest possible sum.

(A) 4410 (B) 16072 (C) 4038090 (D) 4040099 (E) there is none.

Problem 18. 2009 kittens are sleeping in their individual crates, numbered 1 through 2009. A veterinary assistant decides to open all 2009 doors in order while the cats are asleep. He then goes back to the beginning and closes all the even-numbered doors. He then goes back to the beginning and changes every door divisible by three (i.e., if it's open, he closes it, and if it's closed, he opens it). He then continues this process for every integer $k \leq 2009$, so that on the last trip, he changes precisely the 2009^{th} door. When the kittens awake in the morning, what is the largest numbered crate whose kitten will roam free?

(A) 1 (B) 1024 (C) 1936 (D) 2009 (E) None of the above

Problem 19. Rachelle has 100 letters addressed to 100 different people and must place them in corresponding envelopes. Out of boredom, she puts one letter at random in each envelope. What is the expected number of letters that end up in correct envelopes?

(A) 0 (B) $1/e$ (C) 1 (D) 2 (E) 3

Problem 20. What is the product of the lengths of all the diagonals of a regular octagon with sidelength 1? (A diagonal is a line segment connecting two vertices that are not adjacent.)

(A) $\dfrac{(2+\sqrt{2})^{10}}{1024}$ (B) $256(2+\sqrt{2})^4$ (C) $\dfrac{(2+\sqrt{2})^{14}}{4}$ (D) $\dfrac{(2+\sqrt{2})^{28}}{16}$ (E) None of the above

Problem 21. What is the least number n so that a 30°–30°–120° triangle can be cut into n acute triangles?

(A) 8 (B) 9 (C) 10 (D) 11 (E) None of the above

Problem 22. What is the largest number of knights that may be placed on a toroidal 5×5 chessboard so that no knight attacks another? A *toroidal* chessboard is one in which the left and right edges have been identified, as well as the top and bottom edges. Thus a piece can move off the left end of the board and end up at the same height on the right end, and similarly with top and bottom.

(Recall that in chess, a *knight* can move two squares horizontally and one square vertically, or two squares vertically and one square horizontally. The knight, however, does not attack the squares along the way to its destination; thus a single knight attacks 8 squares.)

In the following picture, the knight is denoted by N and it attacks the eight numbered squares.

			8	5
6		7		
3		2		
			1	4
	N			

(A) 4 (B) 5 (C) 6 (D) 7 (E) None of the above

Problem 23. What is the 57^{th} digit in the decimal expansion of $1/23$?

(A) 0 or 9 (B) 1 or 8 (C) 2 or 7 (D) 3 or 6 (E) 4 or 5

Problem 24. In order to win a (tennis-like) game, one must win 3 points and also win by a margin of 2 points. (Thus, possible winning scores are 3-0, 3-1, 4-2, 5-3, etc.) If Boris wins each point with probability $p = 2/3$, what is the probability that he wins the game?

(A) 176/405 (B) 304/405 (C) 112/135 (D) 132/135 (E) None of the above

Problem 25. A set S of (distinct) positive integers has the property that the sum of any three of them is a prime number. What is the largest possible number of elements S can have?

(A) 3 (B) 4 (C) 5 (D) 6 (E) None of the above

Authors. Written by Boris and Valery Alexeev (4,5,7,9,10,15,22 originals)+ (1,2,3,8,11,12,21,25 borrowed), and by Alex Rice, Mo Hendon and Ted Shifrin (6,13, 14,16,17,18,19,20,23,24).