Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

November 16, 2013

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, skip a space, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

Problem 1. Find $\prod_{n=2}^{2013}\left(1-\frac{1}{n}\right)$. The notation "$\prod_{n=1}^{k} a_n$" means $a_1 \cdot a_2 \cdot a_3 \cdots a_k$.

(A) $1-\frac{1}{2014}$ (B) $\frac{1}{2013}$ (C) $1+\frac{1}{2013}$ (D) $\frac{1}{2013!}$ (E) $1-\frac{1}{2013!}$

Problem 2. If a triangle with a 30° angle is inscribed in a circle of radius 1, how long is the side opposite the 30° angle?

(A) 1 (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\sqrt{3}$ (E) it depends on the other angles in the triangle

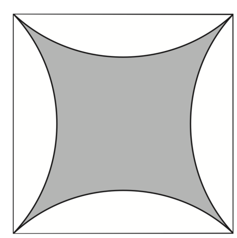
Problem 3. If you travel a certain distance at a rate of R_1 miles/hr, and the same distance again at a rate of R_2 miles/hr, what is your average rate in miles per hour for the whole trip?

(A) $\frac{R_1+R_2}{2}$ (B) $\frac{1}{2}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)$ (C) $\frac{1}{2}\frac{1}{\frac{1}{R_1}+\frac{1}{R_2}}$ (D) $\frac{2}{\frac{1}{R_1}+\frac{1}{R_2}}$ (E) None of the above

Problem 4. What is the smallest positive integer n so that the leftmost digit of $(11)^n$ is **not** 1?

(A) 7 (B) 8 (C) 9 (D) 10 (E) there is no such integer

Problem 5. In the diagram shown at right, 4 circular arcs pass through the corners of a 1 by 1 square. Each of the arcs is tangent to the diagonals of the square. What is the area of the shaded region?



(A) $1-\frac{\pi}{4}$ (B) $1+\sqrt{3}-\frac{2\pi}{3}$ (C) $2-\frac{\pi}{2}$ (D) $4-\pi$ (E) $\frac{\pi}{2}-1$

Problem 6. What is the smallest natural number r so that no number of the form $n!$ ends in exactly r zeroes?

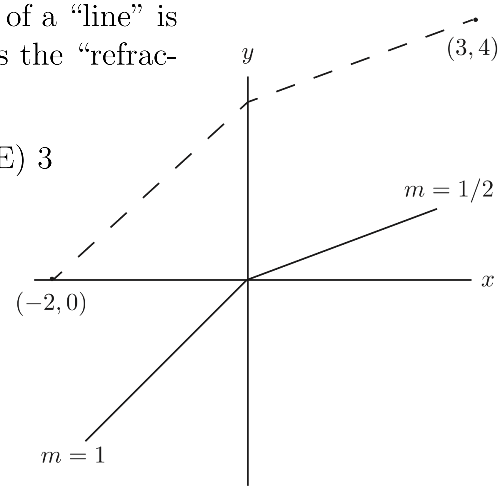
(A) 4 (B) 5 (C) 6 (D) 10 (E) 25

Problem 7. If $\tan(x)+\tan(y)=2013$ and $\tan(x+y)=2014$, what is $\cot(x)+\cot(y)$?

(A) $\frac{1}{2013}$ (B) $\frac{2013}{2014}$ (C) $2013\cdot 2014$ (D) 2013 (E) 2014

Problem 8. In "refraction geometry", the slope of a "line" is halved as the line crosses the y-axis. Where does the "refraction line" from $(-2, 0)$ to $(3, 4)$ cross the y-axis?

(A) 1 (B) 8/5 (C) 2 (D) 16/7 (E) 3



Problem 9. Suppose that n is the largest integer for which 3^n divides the number

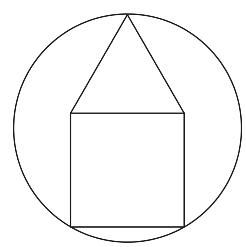
$$1234567891011121314\ldots2013$$

obtained by concatenating the decimal digits of the positive integers $1, 2, 3, \ldots, 2013$. Find n.

(A) 0 (B) 1 (C) 2 (D) 3 (E) None of the above

Problem 10. A square of side length 1 is topped with an equilateral triangle, then inscribed in a circle as shown. What is the radius of the circle?

(A) 1 (B) $\frac{1}{2} + \frac{\sqrt{3}}{4}$ (C) $\frac{3}{2}$ (D) $2 + \sqrt{3}$ (E) 2



Problem 11. How many positive integers n are there for which $\lfloor \frac{n^2}{5} \rfloor$ is a prime number? Here the notation $\lfloor x \rfloor$ means the largest integer not exceeding x.

(A) 2 (B) 3 (C) 4 (D) 5 (E) infinitely many

Problem 12. Let $\theta_1, \theta_2, \theta_3, \theta_4$ be the four complex roots of the polynomial

$$P(x) = 20x^4 + 13x^3 + 11x + 16.$$

Find the numerical value of

$$(1+\theta_1)(1+\theta_2)(1+\theta_3)(1+\theta_4).$$

(A) 16 (B) -13 (C) 12 (D) $\frac{4}{5}$ (E) $\frac{3}{5}$

Problem 13. The function $B(n)$ satisfies

$$B(0) = 0, \quad B(2n) = B(n) \quad \text{and} \quad B(2n+1) = B(n) + 1$$

for every nonnegative integer n. Find $B(2013)$.

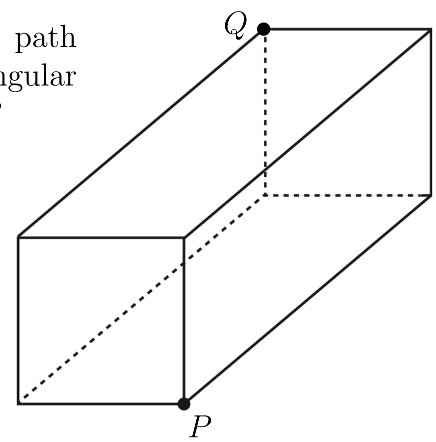
(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Problem 14. Consider a triangular array of numbers whose nth row, $1 \leq n \leq 100$, consists of the number n repeated n times. What is the average of all of the numbers in the array?

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 2 & & 2 & & & & \\
 & & & 3 & & 3 & & 3 & & & \\
 & & 4 & & 4 & & 4 & & 4 & & \\
 & 5 & & 5 & & 5 & & 5 & & 5 & \\
\iddots & & \iddots & & & & & & \ddots & & \ddots \\
100 & & & & & \cdots & & & & & 100
\end{array}$$

(A) $\frac{101}{2}$ (B) $\frac{20301}{6}$ (C) 5050 (D) $\frac{2030100}{6}$ (E) $\frac{201}{3}$

Problem 15. What is the length of the shortest path lying entirely on the surface of a 1 by 1 by 2 rectangular box, going from one corner to the opposite corner?



(A) $\sqrt{6}$ (B) $\sqrt{8}$ (C) $\sqrt{10}$ (D) $2+\sqrt{2}$ (E) 4

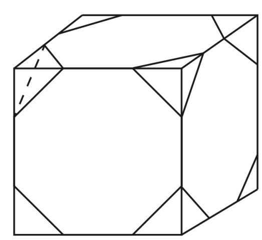
Problem 16. Find the sum of the infinite series

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9}+\frac{1}{12}+\ldots,$$

where the denominators are all of the positive integers which have no prime factors larger than 3.

(A) 8/3 (B) e (C) 3 (D) π (E) the sum does not converge

Problem 17. Start with a 1 by 1 by 1 cube. Cut a tetrahedron off of each corner in such a way that the faces of the resulting polyhedron are all equilateral triangles or regular octagons. What is the edge length for each of those faces?

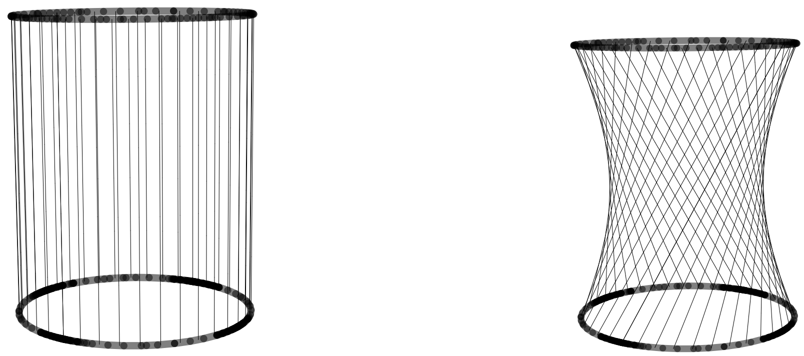


(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2-\sqrt{2}}{2}$ (D) $\sqrt{2}-1$ (E) None of the above

Problem 18. Two urns each contain the same number $N>0$ of balls. Each ball is either red or black, and there are balls of both colors in each urn. From each urn we randomly choose a ball, note its color, return it to the urn, then choose and note again. Now suppose that the probability of choosing 2 red balls from the first urn equals the probability of choosing 2 red balls or 2 black balls from the second urn. What is the smallest N can be?

(A) 2 (B) 5 (C) 7 (D) 12 (E) it is not possible for those probabilities to be equal

Problem 19. Construct a cylinder as follows: Begin with two circles of radius r and join them by strings of length $h > 2r$ as shown. Now rotate the top circle $90°$ around the central axis of the cylinder. What is the radius of the "waist" of the resulting hyperboloid?



(A) r (B) $r/\sqrt{2}$ (C) $\sqrt{2}r$ (D) $\sqrt{\frac{1}{2}r^2 + h^2}$ (E) $\sqrt{h^2 - 2r^2}$

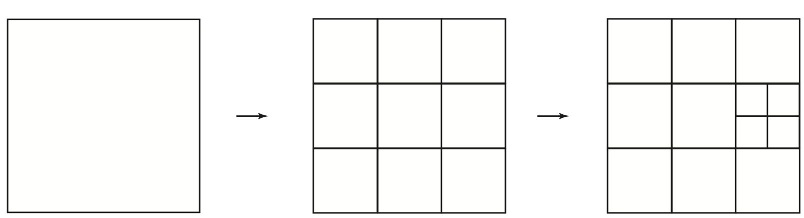
Problem 20. Suppose $f(x)$ is a polynomial of degree 4 with integer coefficients and $f(2013) = f(2014) = 1$. What is the largest number of integer solutions that $f(x) = 0$ can have?

(A) 0 (B) 1 (C) 2 (D) 3 (E) 4

Problem 21. Begin with a square. You may, if you choose, partition it into squares by drawing lines parallel to the sides of the squares. All such partitioning lines must completely cross, but not extend beyond, the square being partitioned. You now have n^2 squares, for some natural number n. Notice that a square, once partitioned, is no longer included in the count.

You may now, if you choose, partition any of the squares in your partitioned square. And you may, if you choose, continue to partition squares for any finite number of steps.

Any such construction determines a certain number of squares, counting only squares which are not further subdivided. How many positive integers **do not** arise this way?



(A) 7 (B) 8 (C) 9 (D) 10 (E) infinitely many

Problem 22. Let ℓ_1 be the line $y = x$ and ℓ_2 the line $y = 2x$. Let

$$C = \{p \in \mathbb{R}^2 : d(p, \ell_1) + d(p, \ell_2) = 1\}.$$

Find the area of the region enclosed by C. **Note:** $d(p, \ell_m)$ is the perpendicular distance from the point p to the line ℓ_m.

(A) 10π (B) $2\sqrt{10}\pi$ (C) 1 (D) $\sqrt{40}$ (E) $\sqrt{20 + 6\sqrt{10}}$

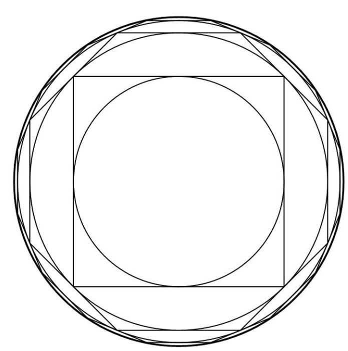
Problem 23. If $a_0 = 0$, $a_1 = 1$, and $a_{n+1} = 1 + \frac{a_n^2 - a_n a_{n-1} + 1}{a_n - a_{n-1}}$ for $n = 1, 2, 3, 4, \ldots$, find the integer closest to a_{10}.

(A) 4 (B) 11 (C) 15 (D) 16 (E) 24

Problem 24. A classic, from Sam Loyd: Mary and Ann's ages add up to 44 years, and Mary is twice as old as Ann was when Mary was half as old as Ann will be when Ann is three times as old as Mary was when Mary was three times as old as Ann. How old is Ann?

(A) $13\ \frac{1}{2}$ (B) $14\ \frac{1}{2}$ (C) $15\ \frac{1}{2}$ (D) $16\ \frac{1}{2}$ (E) $17\ \frac{1}{2}$

Problem 25. Let C_1 be a circle of radius 1, and P_1 a square circumscribed around C_1. For each $n \geq 2$, let C_n be the circle of radius r_n circumscribed around P_{n-1}, and let P_n be a regular 2^{n+1}-gon circumscribed around C_n. Find $\lim_{n\to\infty} r_n$.



(A) $\frac{1+\sqrt{5}}{2}$ (B) 2 (C) e (D) π (E) $\frac{\pi}{2}$