



Sponsored by: UGA Math Department and UGA Math Club

WRITTEN TEST, 25 PROBLEMS / 90 MINUTES
October 22, 2016

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Let a represent a digit from 1 to 9. Which a gives

$$\frac{a!}{aa + a^2} = 2016?$$

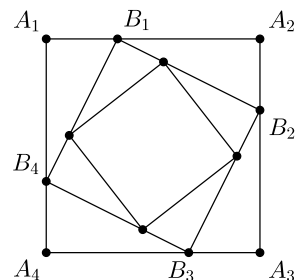
Here aa indicates concatenation of the digit a . For example, if $a = 1$, then $aa = 11$.

- (A) 5 (B) 6 (C) 7 (D) 8 (E) 9

Problem 2. Suppose p and q are positive integers and $p < q$. How many of the following must be true? i) $p \leq q$ ii) $p < q + 1$ iii) $p \leq q + 1$ iv) $p + 1 < q$ v) $p + 1 \leq q$

- (A) one (B) two (C) three (D) four (E) five

Problem 3. Start with a square with vertices A_1, A_2, A_3, A_4 . Let B_1 be the point $1/3$ of the way from A_1 to A_2 , and similarly for B_2, B_3 , and B_4 (see figure). Repeat this construction on the quadrilateral $B_1B_2B_3B_4$ to construct another quadrilateral $C_1C_2C_3C_4$. What is the ratio of the area of $A_1A_2A_3A_4$ to the area of $C_1C_2C_3C_4$?



- (A) 4 (B) $\frac{9}{5}$ (C) $\frac{81}{25}$ (D) $\frac{9}{4}$ (E) 9

Problem 4. What is the 100th digit in the decimal expansion of $\frac{1}{7}$?

- (A) 0 or 9 (B) 1 or 8 (C) 2 or 7 (D) 3 or 6 (E) 4 or 5

Problem 5. What are the leftmost 3 digits of

$$142857143 \cdot 121935032 \quad ?$$

- (A) 168 (B) 171 (C) 172 (D) 173 (E) 174

Problem 6. Compute

$$\frac{1}{\sqrt{0} + \sqrt{2}} + \frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{2} + \sqrt{4}} + \cdots + \frac{1}{\sqrt{2014} + \sqrt{2016}}.$$

(A) $\frac{\sqrt{2016}}{2}$ (B) $\frac{\sqrt{2016} + \sqrt{2015}}{2}$ (C) $\frac{\sqrt{2016} + \sqrt{2014}}{2}$
(D) $\frac{\sqrt{2016} + \sqrt{2015} - \sqrt{2014}}{2}$ (E) $\frac{\sqrt{2016} + \sqrt{2015} - 1}{2}$

Problem 7. The roots of the 6th degree polynomial

$$x^6 - 19x^5 + 145x^4 - 565x^3 + 1174x^2 - 1216x + 480$$

are 1, 2, 3, 4, 5, one of which occurs with multiplicity 2. Which one?

- (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

Problem 8. How many numbers n , $0 < n < 10^5$, have the property that any permutation of their digits is divisible by 5?

- (A) 4 (B) 5 (C) 31 (D) 83 (E) 1999

Problem 9. If you know P is closer to $(3, 1)$ than to $(1, 3)$ and closer to $(2, 4)$ than to $(4, 6)$, and closer to $(3, 2)$ than to $(3, 0)$, what is the area of the region in which P is constrained to lie?

- (A) 9 (B) 18 (C) 27 (D) 0, because there is no such point P
(E) ∞ , because the region is unbounded

Problem 10. Four friends go on vacation, each to a different location. Each sends a postcard to two friends, chosen at random from the three others with equal probability. What is the probability that each one gets postcards from the same two friends she sent postcards to?

- (A) 0 (B) $\frac{1}{81}$ (C) $\frac{1}{27}$ (D) $\frac{1}{8}$ (E) $\frac{1}{6}$

Problem 11. Seven friends go on vacation, each to a different location. Each sends a postcard to three of the friends, chosen at random from the six others with equal probability. What is the probability that each one gets postcards from the same three friends he sent postcards to?

- (A) 0 (B) $\frac{1}{\binom{6}{3}^4}$ (C) $\frac{1}{\binom{6}{3}^3}$ (D) $\frac{1}{6^3}$ (E) $\frac{1}{6!}$

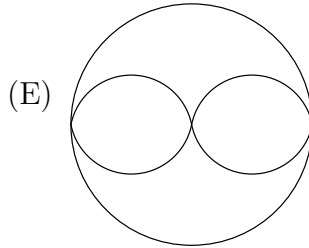
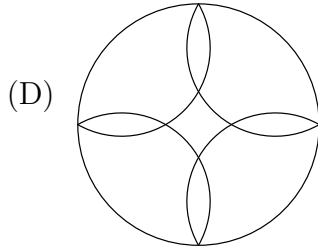
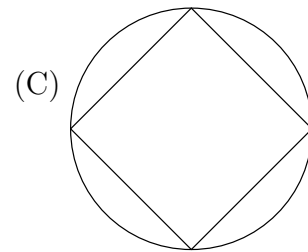
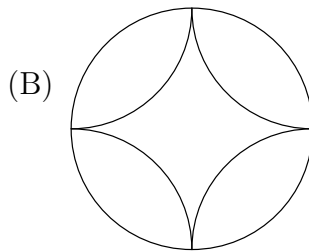
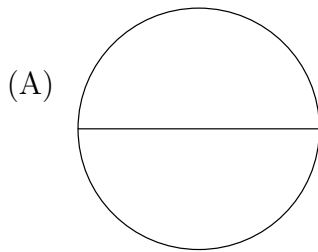
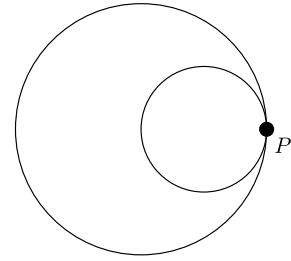
Problem 12. There are 100 lockers in a row, labelled 1-100, and all the doors are initially closed. One hundred mischievous students begin opening and closing doors: the first student opens every door; the second closes every other door, beginning with the first; \dots , the n th student changes the status of every n th door, beginning with the first. How many of the doors will be open after the 100th student finishes?

- (A) 1 (B) 9 (C) 10 (D) 11 (E) 50

Problem 13. In a scalene triangle with side lengths $a < b < c$ opposite angles $\alpha < \beta < \gamma$, which of the following is largest?

- (A) $\frac{a}{\sin \alpha}$ (B) $\frac{b}{\sin \beta}$ (C) $\frac{c}{\sin \gamma}$ (D) the diameter of the circumscribed circle
(E) those are all the same

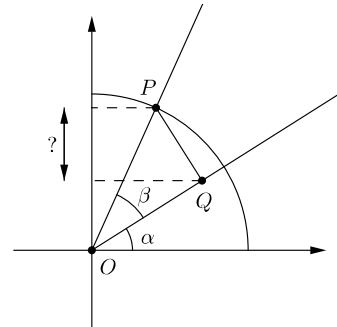
Problem 14. A circle of radius 1 rolls without slipping inside a circle of radius 2. If P is a point on the circumference of the smaller circle, what is the shape of the path that P traces as the inner circle rolls?



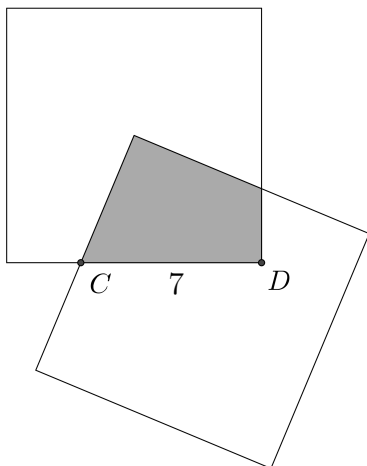
Problem 15. Suppose x and y are two-digit numbers with the property that $\frac{1}{x} = 0.\overline{0y}$ and $\frac{1}{y} = 0.\overline{0x}$. (For example, if $x = 12$, then $\frac{1}{y} = 0.\overline{012} = 0.012012012\dots$) What is $x + y$?

- (A) 64 (B) 70 (C) 120 (D) There is more than one pair.
 (E) There are no such x and y .

Problem 16. Two angles α and β are chosen in the first quadrant as shown, together with a circle of radius 1. P lies on the circle, and $PQ \perp OQ$. What is the vertical distance between P and Q ?

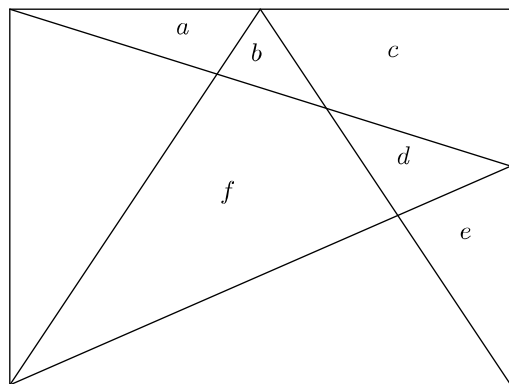


- (A) $\sin(\beta)$ (B) $\sin(\alpha + \beta)$ (C) $\sin(\beta - \alpha)$ (D) $\sin(\beta) \cos(\alpha)$ (E) $\sin(\alpha) \cos(\beta)$



Problem 17. Consider two equal squares of side length 10. Assume a vertex of one of the two squares coincides with the center of the other square. What is the area of the intersection of these squares?

- (A) $\frac{100}{3}$ (B) 20 (C) $\frac{10\sqrt{2}}{2}$ (D) 25 (E) $8\sqrt{2}$.



Problem 18. In the diagram shown, the letters a – f represent the areas of the regions. Which combination of a, b, c, d, e adds up to f ?

- (A) $a + b + c + d + e$ (B) $a + c + e$ (C) $a + b + d + e$ (D) $b + c + d$
 (E) no combination is guaranteed to equal f

Problem 19. Think of a positive integer x . Multiply it by 3. Add up the digits. Multiply that by 3. Add the digits one last time. Is your number now 9? Let n be the smallest x such that your number at the end is not 9. What is the sum of the digits of n ?

- (A) 5 (B) 6 (C) 8 (D) 9 (E) 11

Problem 20. Start with a cube. Connect the centers of the faces to create an octahedron. Connect the centers of the faces of the octahedron to form another cube. What is the ratio of the volume of the larger cube to the volume of the smaller cube?

- (A) 6 (B) 8 (C) 9 (D) 27 (E) 36

Problem 21. Let $r_1, r_2, r_3,$ and r_4 be real roots of the polynomial

$$x^8 - 6x^7 - 15x^4 - 6x + 1$$

with $r_1 \leq r_2 \leq r_3 \leq r_4$. What is $r_1r_2 + r_3r_4$?

- (A) -5 (B) 0 (C) 2 (D) 3 (E) 5

Problem 22. We all know that if a and b are the legs of a right triangle and c is the hypotenuse, then $a^2 + b^2 = c^2$. If instead we consider the equation

$$a^{-2} + b^{-2} = x^{-2},$$

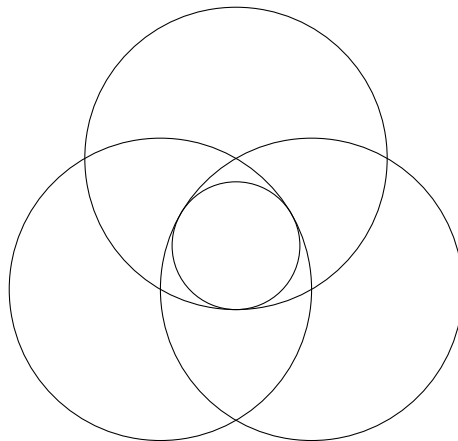
what does x represent?

- (A) the hypotenuse of the triangle (B) the perimeter
(C) the altitude (from hypotenuse to opposite vertex) (D) the area
(E) the diameter of the inscribed circle

Problem 23. Each point (p, q) in the unit square $0 \leq p \leq 1, 0 \leq q \leq 1$ corresponds to a parabola $y = x^2 + px + q$. Let R be the region in the unit square corresponding those parabolas that intersect the line $y = x$. Which of the following is true about the area A of R ?

- (A) $0 \leq A < \frac{1}{5}$ (B) $\frac{1}{5} \leq A < \frac{2}{5}$ (C) $\frac{2}{5} \leq A < \frac{3}{5}$ (D) $\frac{3}{5} \leq A < \frac{4}{5}$
(E) $\frac{4}{5} \leq A \leq 1$

Problem 24. Let three circles of radius 1 each intersect the others' centers and draw a small circle internally tangent to all three as in the figure below. What is the radius of the small circle?



- (A) $\frac{1}{3}$ (B) $1 - \frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{\sqrt{5}-1}{3}$ (E) $\frac{1}{6} + \frac{1}{2\sqrt{3}}$

Problem 25. Amber the ant and Byron the beetle have homes on opposite vertices of a unit cube. One day Amber and Byron arrange to meet. Both set off from their homes at the same time. Each walks along edges from vertex to vertex at a constant speed of one edge per minute, and each time one reaches a vertex they choose at random a neighboring edge to walk along (the choice might be to return along the edge they just took). What is the expected number of minutes it will take for them to find each other?

- (A) 5 (B) 7 (C) 10 (D) 13 (E) they never meet