Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 21, 2017

Instructions

1. At the top of the left of side 1 of your scan-tron answer sheet, fill in your last name, fill in your first name, and then bubble in both appropriately. Below the name, in the center, **fill in your 4-digit Identification Number and bubble it in.**

2. This is a 90-minute, 25-problem exam.

3. Scores will be computed by the formula

$$10 \cdot C + 2 \cdot B + 0 \cdot I \ ,$$

   where C is the number of questions answered correctly, B is the number left blank, and I the number of questions answered incorrectly. Random guessing will not, on average, improve one's score.

4. No calculators, slide rules, or any other such instruments are allowed.

5. Scratchwork may be done on the test and on the three blank pages at the end of the test. Credit will be given only for answers marked on the scan-tron sheet.

6. If you finish the exam before time is called, turn in your scan-tron sheet to the person in the front and then exit quietly.

7. If you need another pencil, more scratch paper, or require other assistance during the exam, raise your hand.

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

Problem 1. Suppose $a_n > 0$ for every integer n, and that $a_{n+1} = a_n^2 - 2^n$. If $a_4 = 2017$, what is a_0?

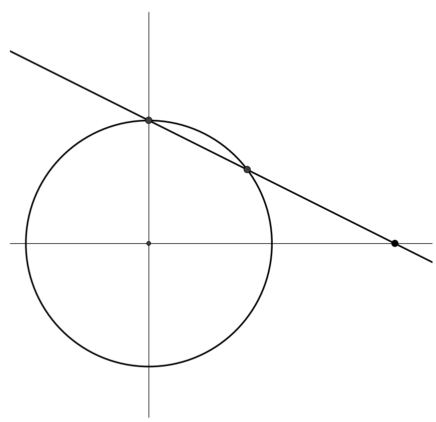
(A) 5 (B) 4 (C) 3 (D) 2 (E) 1

Problem 2.

Of A and B this is the lore:
When added they make 24.
If A over 3
is A over B,
what's $A + 2B$ plus 2 more?

(A) 28 (B) 29 (C) 29 or 50 (D) 47 (E) 50

Problem 3. Draw the circle $x^2+y^2 = 1$, then draw the line through the "north pole" $(0, 1)$ meeting the x-axis at $(2, 0)$. What is the x-coordinate of the other point where the line meets the circle?



(A) $\frac{4}{5}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{2}{3}$ (E) 1

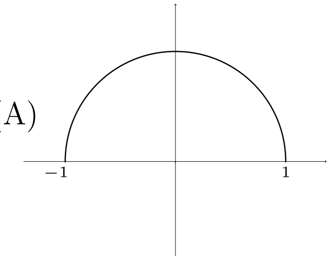
Problem 4. In this magic square, there is exactly one way to fill the empty squares so that every row, every column, and both main diagonals add up to the same value. What is that value?

		1
3	2	

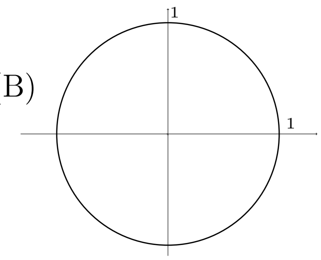
(A) 15/2 (B) 9 (C) 21/2 (D) 12 (E) 15

Problem 5. Which of the following is the graph of $\cos(y) - \sin(x) = 0$?

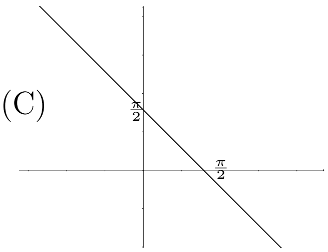
(A)



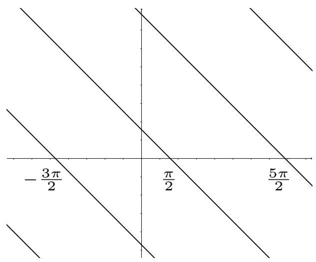
(B)



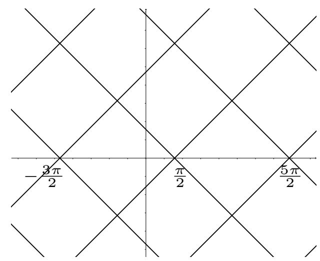
(C)



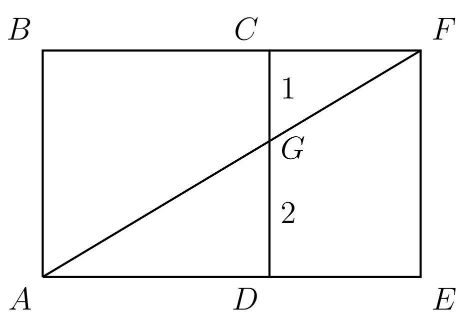
(D)



(E)



Problem 6. Consider the rectangle $ABFE$ as shown in the figure below. $ABCD$ is a square. If $CG = 1$ and $GD = 2$, then what is the perimeter of the rectangle $ABFE$?



(A) $6 + 2\sqrt{5}$ (B) 10 (C) 12 (D) 15 (E) 24

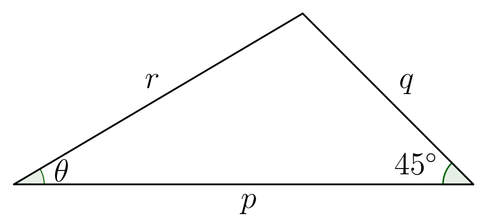
Problem 7. Let $f(x)$ and $g(x)$ be real-valued functions defined for all real numbers x. Suppose that for certain constants a, b, A, B, we have that

$$0 < a \leq f(x) \leq A, \quad \text{and} \quad 0 < b \leq g(x) \leq B$$

for all real numbers x. Which of the following must also be true for all x?

(A) $\frac{a}{b} \leq \frac{f(x)}{g(x)} \leq \frac{A}{B}$, (B) $\frac{a}{B} \leq \frac{f(x)}{g(x)} \leq \frac{A}{b}$, (C) $\frac{A}{b} \leq \frac{f(x)}{g(x)} \leq \frac{a}{B}$,
(D) $\frac{A}{B} \leq \frac{f(x)}{g(x)} \leq \frac{a}{b}$, (E) none of these must be true

Attention! The following 5 questions (#8 – #12) all concern the functions $c(\theta)$ and $s(\theta)$, which are defined as follows. Given an angle θ, which we will measure in degrees, construct a triangle with angles θ and $45°$, and side lengths as indicated in the diagram.



Define $c(\theta) = \dfrac{p}{r}$ and $s(\theta) = \dfrac{q}{r}$.

Problem 8. Find $c(60°)$.

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{2}}{2}$ (C) $\frac{\sqrt{3}}{2}$ (D) 1 (E) $\frac{1+\sqrt{3}}{2}$

Problem 9. What is the domain of $c(\theta)$?

(A) $[0°, 360°)$ (B) $(0°, 360°)$ (C) $(0°, 90°)$ (D) $(0°, 135°)$ (E) all real numbers

Problem 10. What is the range of $s(\theta)$?

(A) $[-1, 1]$ (B) $[0, 1]$ (C) $(0, 1)$ (D) $(0, \sqrt{2})$ (E) $(0, \sqrt{2}]$

Problem 11. If $c(\theta) = 1$, what is θ?

(A) $0°$ (B) $30°$ (C) $45°$ (D) $60°$ (E) $90°$

Problem 12. The curve in the plane parametrized by $(c(\theta), s(\theta))$ is part of which kind of curve?

(A) a sine curve (B) a circle (C) an ellipse that is not a circle (D) a hyperbola (E) a line

Problem 13. Suppose a and b are nonzero decimal digits (1–9), with the property that

$$(aa)^2 + (bb)^2 = aabb.$$

What is $a + b$?

(A) 8 (B) 10 (C) 11 (D) 16 (E) there are no such numbers

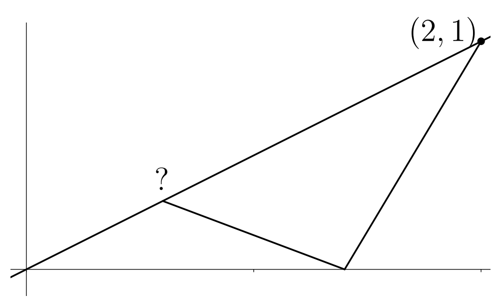
Problem 14. a and b are lines in the plane. If you translate a to the left 3 units, it coincides with b. If instead you translate a up by 4 units, it coincides with b again. What is the distance between a and b?

(A) 2 (B) $\frac{12}{5}$ (C) $\sqrt{\frac{12}{7}}$ (D) 2.5 (E) $\frac{1+\sqrt{5}}{2}$

Problem 15. How many ordered pairs of positive integers (x, y) satisfy $x \cdot y \leq 100$.

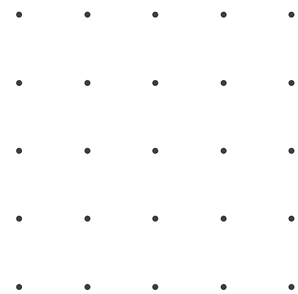
(A) 432 (B) 482 (C) 532 (D) 572 (E) 582

Problem 16. What is the length of the shortest path that starts at $(2, 1)$, touches the x-axis, then returns to some point on the line $y = \frac{1}{2}x$.



(A) 1 (B) $\sqrt{5}$ (C) 2 (D) $\frac{\sqrt{80}}{5}$ (E) $\frac{\sqrt{80}+5}{5}$

Problem 17. Consider a 5×5 grid of points, as pictured at right. How many squares can be drawn with all four corners on the grid?



(A) 30 (B) 40 (C) 48 (D) 49 (E) 50

Problem 18. Let

$$S = 3 + 33 + 333 + \ldots + \underbrace{333 \cdots 333}_{32\ 3\text{'s}}.$$

Find the sum of the digits of S.

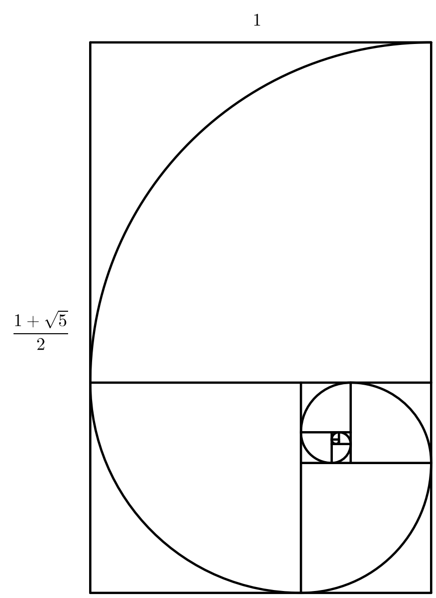
(A) 189 (B) 153 (C) 108 (D) 99 (E) 135

Problem 19. Recall that a "golden rectangle" has the following properties:

(1) The ratio of the long side to short side is $g = \frac{1+\sqrt{5}}{2}$.

(2) If you draw a line separating the rectangle into a square and a smaller rectangle, then the smaller rectangle is similar to the original rectangle.

Now draw a quarter circle in each square as shown. How long is this "golden spiral"?



(A) $\dfrac{\pi(1+\sqrt{5})}{4}$ (B) $\dfrac{\pi(3+\sqrt{5})}{4}$ (C) $\dfrac{\pi(1+3\sqrt{5})}{4}$ (D) $\dfrac{\pi(3+3\sqrt{5})}{4}$ (E) infinity

Problem 20. Find the exact value of $\cos(\pi/7)\cos(2\pi/7)\cos(3\pi/7)$.

(A) $\frac{1}{8}$ (B) $\frac{\sqrt{2}}{8}$ (C) $\frac{\sqrt{3}}{8}$ (D) $\frac{\sqrt{5}}{8}$ (E) $\frac{\sqrt{6}}{8}$

Problem 21. Let $a_0 = 2$, $a_1 = 1$, $a_2 = 2$, and for $n \geq 2$ define

$$a_{n+1} = \frac{a_n + a_{n-1} + a_{n-2}}{3}.$$

Find $\lim_{n\to\infty} a_n$.

(A) $\frac{\sqrt{5}-1}{2}$ (B) $\sqrt{2}$ (C) $\frac{3}{2}$ (D) $\frac{1+\sqrt{5}}{2}$ (E) $\frac{5}{3}$

Problem 22. Two children are playing on two toy pianos. Each toy piano has 5 notes. Every second each child switches at random from hitting the current note to a different but neighboring note. If the children start at random notes, what is the probability that they will eventually play the same note at the same time?

(A) $\frac{12}{25}$ (B) $\frac{13}{25}$ (C) $\frac{3}{5}$ (D) $\frac{4}{5}$ (E) 1

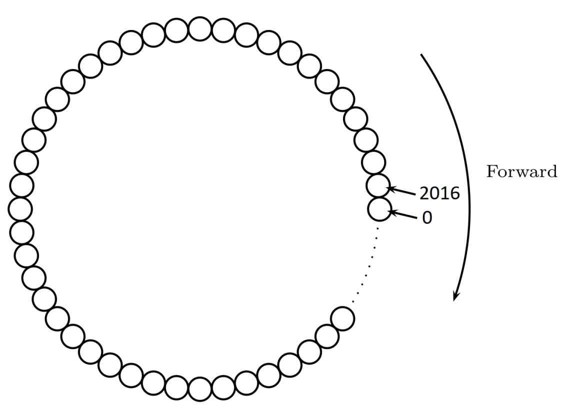
Problem 23. The functions e^{1-x} and $-\ln(x-1)$ intersect only at the point (p, q). What is $q - p$?

(A) $-\ln 2$ (B) $-\frac{e}{3}$ (C) -1 (D) $-\frac{4}{3}$ (E) $-\frac{e}{2}$

Problem 24. For how many positive integers $n \leq 5^5$ does 5 not divide $\binom{2n}{n}$?

(A) 243 (B) 256 (C) 625 (D) 2401 (E) 2500

Problem 25. A circular arrangement of 2017 lily pads floats in Shifrin pond. The lily pads are consecutively numbered 0 to 2016, with lily pad 2016 next to lily pad 0. Ted the Toad can jump forward n^2 steps, for any positive integer n, wrapping around the circle as necessary. (For example, starting from lily pad 0, he can reach lily pad 8 in one jump of length $45^2 = 2025$.) What is the smallest positive integer d such that, starting from lily pad 0, Ted can reach any other lily pad within d jumps?



(A) 2 (B) 3 (C) 4 (D) 5 (E) 6