1785

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
November 18, 2006

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. It takes David 6 hours to paint his fence. Since he doesn't have enough time, he asks his friends Alex and Chris to help. If Alex can paint the entire fence in just 3 hours and Chris can paint the entire fence in 4 hours, how many hours will it take all three to paint the fence?

Answer. 4/3 (1 1/3)

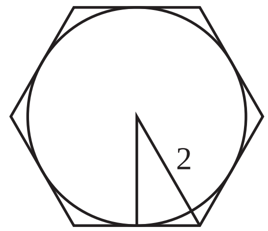
Solution. In one hour, David paints 1/6 of the fence, Alex paints 1/3 of the fence, and Chris paints 1/4 of the fence. So, all three working together paint 3/4 of the fence in one hour. It takes them 4/3 hours to paint the entire fence.

Problem 2. A circle is inscribed in a regular hexagon. If the perimeter of the hexagon is 12, what is the area of the circle?

Answer. 3π

Solution. Each side of the hexagon is 2, and so the radius of the circle is

$\sqrt{3}$. Thus, the area of the circle is 3π.



Problem 3. How many points (m, n) with integer coordinates are on the line segment joining $(-2, 3)$ and $(34, 30)$?

Answer. 10

Solution. The line joining $(-2, 3)$ and $(34, 30)$ has slope $\dfrac{30-3}{34-(-2)} = \dfrac{3}{4}$. Thus, starting at any point with integer coordinates, if we move horizontally 4 units and vertically 3 units, we will come to the next point on the line with integer coordinates. Since $36/4 = 9$, we have our original point and 9 other points on the line segment.

Problem 4. Four identical tennis balls are packed tightly in a cylindrical can. What fraction of the volume of the can is unoccupied?

Answer. 1/3

Solution. If the radius of the can is r, then its height is $8r$, and the volume of the cylinder is $8\pi r^3$. The volume occupied by the four balls is $4 \cdot \dfrac{4}{3}\pi r^3 = \dfrac{16}{3}\pi r^3$. Thus, the fraction unoccupied is

$$\frac{8 - 16/3}{8} = \frac{1}{3}.$$

Problem 5. What is the angle, in degrees, formed by the hands of a clock at precisely 1:20? (Choose the angle less than 180°.)

Answer. 80

Solution. In each hour, the hour hand moves through $\frac{360°}{12} = 30°$. So, in 1 1/3 hours, the hour hand makes an angle of 40° with the vertical, whereas the minute hand makes an angle of $\frac{360}{3} = 120°$ degrees with the vertical. Thus, the angle between the two hands is 80°.

Problem 6. Fill in the missing digits so that N will be divisible by 99:

$$N = 8\_\_5\,2\_\_6$$

Answer.

$$N = 805266$$

Solution. Say $N = 8\,a\,5\,2\,b\,6$. Since N is divisible by 9, we know that $8 + a + 5 + 2 + b + 6 = 21 + a + b$ must be divisible by 9. Similarly, since N is divisble by 11, we know that $(8 + 5 + b) - (a + 2 + 6) = b - a + 5$ must be divisible by 11. Since a and b are integers between 0 and 9 inclusive, it follows that

$$a + b = 6 \text{ or } 15$$
$$-a + b = -5 \text{ or } 6\,.$$

Obviously, $a = 0$ and $b = 6$ gives a solution. But it is the only solution: Adding and subtracting the two equations with various right-hand sides gives $2b = 1$ (6 and -5), $2b = 10$ and $2a = 20$ (15 and -5), and $2b = 21$ (15 and 6).

Problem 7. A 25-meter ladder is placed against the wall and the foot of the ladder is 7 meters away from the wall. When the top of the ladder slides 4 meters down the wall, how far does the foot of the ladder slide (in meters)?

Answer. 8

Solution. The key to this problem is to recognize two Pythagorean triangles: 3-4-5 and 7-24-25. In its original position, the height of the ladder is 24 meters, and when it slides 4 meters down, its height is 20 meters. That means that the foot of the ladder is now 15 meters from the wall, so it has slid a distance of 8 meters.

Problem 8. A fair coin is tossed 8 times. What is the probability that it comes up heads at least 4 times?

Answer. $\frac{163}{256}$

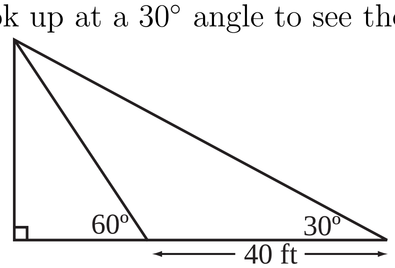
Solution. There are $\binom{8}{k}$ ways of throwing k heads, $0 \leq k \leq 8$. These numbers form the eighth row of Pascal's triangle:

$$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1$$

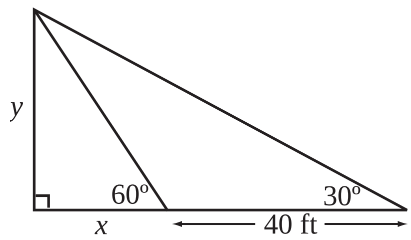
So the answer is

$$\frac{70+56+28+8+1}{2^8} = \frac{163}{256} \left(= \frac{1}{256} \cdot \frac{256+70}{2} \right).$$

Problem 9. An ant on the ground must look up at a 60° angle to see the top of a nearby building. When she walks 40 ft away from the building, she must now look up at a 30° angle to see the top of the building. How high is the building?



Answer. $20\sqrt{3}$ ft



Solution. From the right triangles in the diagram we have

$$\tan 60^\circ = \sqrt{3} = \frac{y}{x} \qquad \text{and} \qquad \tan 30^\circ = \frac{1}{\sqrt{3}} = \frac{y}{x+40}\,.$$

Thus, $x = y/\sqrt{3}$ and $y = \dfrac{x+40}{\sqrt{3}} = \dfrac{y}{3} + \dfrac{40}{\sqrt{3}}$, so $y = 20\sqrt{3}$ ft.

Problem 10. If r and s are the solutions of

$$x^2 + ax + b = 0\,,$$

then express $r^3 + s^3$ in terms of a and b.

Answer. $3ab - a^3$

Solution. Since $x^2 + ax + b = (x-r)(x-s)$, we know that $r + s = -a$ and $rs = b$. Thus,

$$(r+s)^3 = r^3 + 3r^2s + 3rs^2 + s^3 = (r^3 + s^3) + 3rs(r+s)$$

and so $r^3 + s^3 = (-a)^3 - 3b(-a) = 3ab - a^3$.

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