1785

Sponsored by: UGA Math Department and UGA Math Club

CIPHERING ROUND / 2 MINUTES PER PROBLEM
OCTOBER 13, 2007

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. The length of a rectangle is increased by 30% and its width is decreased by 20%. By what percentage does its area increase?

Answer. 4%

Solution. If the original length and width are ℓ and w, respectively, then the new length is 1.3ℓ and the new width is $0.8w$, so the area of the new rectangle is $(1.3\ell)(0.8w) = 1.04\ell w$. Thus, the area increases by 4%.

Problem 2. When two joggers run around a 1-mile oval track in the same direction, one passes the other every 30 minutes. When they run in opposite directions, they pass every 10 minutes. What is the speed of the slower jogger (in mph)?

Answer. 2 mph

Solution. Let x be the speed (in mph) of the faster jogger and y the speed of the slower jogger. Then we have $\frac{1}{2}x = 1 + \frac{1}{2}y$ and $\frac{1}{6}x + \frac{1}{6}y = 1$, so $x - y = 2$ and $x + y = 6$. This gives $y = 2$.

Problem 3. What is the largest integer n so that a regular n-gon with sidelength 1 will fit inside a circle of radius 1?

Answer. 6

Solution. The sides of a regular hexagon inscribed in the unit circle have length 1. Thus, if $n \geq 7$, the sides of a regular n-gon inscribed in the unit circle must be shorter, so, conversely, if we blow up the picture, in order to make the sidelengths 1, the circle must have bigger radius.

Problem 4. If you have 2 black socks and 3 green socks in a drawer, how many socks must you take out in order to be guaranteed a matching pair?

Answer. 3

Solution. This is a trick question, of course. If you pick two socks of different colors initially, no matter what sock you pick the third time, it has to be the same color as one of the first two.

Problem 5. If you have 2 black socks and 3 green socks in a drawer, in the long run, what is the average number of socks you must take out in order to have a matching pair?

Answer. $13/5 = 2.6$

Solution. The probability of picking *two* matching socks is

$$\frac{2}{5} \cdot \frac{1}{4} + \frac{3}{5} \cdot \frac{2}{4} = \frac{2}{5}\,.$$

Thus, we will get a match $2/5$ of the time with *two* socks drawn, and $3/5$ of the time with *three* socks drawn. The expected number of socks we must take out to get a match is

$$2 \cdot \frac{2}{5} + 3 \cdot \frac{3}{5} = \frac{13}{5} = 2.6\,.$$

Problem 6. How many positive factors does the number 288 have (including 1 and 288)?

Answer. 18

Solution. We have $288 = 32 \cdot 9 = 2^5 \cdot 3^2$, so we want to count the numbers of the form $2^j 3^k$, where $0 \leq j \leq 5$ and $0 \leq k \leq 2$. Since there are 6 possible values of j and 3 possible values of k, there are 18 factors.

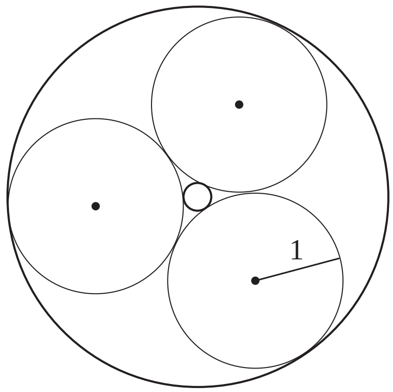
Problem 7. What is the sum of all the positive factors of 288 (including 1 and 288)?

Answer. 819

Solution. We have $144 = 32 \cdot 9 = 2^5 \cdot 3^2$, so we want to sum all numbers of the form $2^j 3^k$, where $0 \le j \le 5$ and $0 \le k \le 2$. This is just the sum

$$(1 + 2 + 2^2 + 2^3 + 2^4 + 2^5)(1 + 3 + 3^2) = \frac{2^6 - 1}{2 - 1} \cdot \frac{3^3 - 1}{3 - 1} = 63 \cdot 13 = 819\,.$$

Problem 8. The three middle circles in the figure have radius 1. What is the sum of the radii of the smallest and largest circles?



Answer. $\frac{4}{3}\sqrt{3}$

Solution. The center of the diagram is the centroid of the triangle formed by the centers of the three middle circles. If the radius of the innermost circle is r and the radius of the outermost circle is R, we have $R = r + 2$, so $r + R = 2(r + 1)$. Since the centroid of a triangle is 2/3 the way down any of its medians, we have $r + 1 = \frac{2}{3}\sqrt{3}$, so $2(r + 1) = \frac{4}{3}\sqrt{3}$.

Problem 9. How many integers between 7 and 2007 inclusive are divisible by neither 3 nor 5?

Answer. 1067

Solution. There are $\frac{2007 - 9}{3} + 1 = 667$ multiples of 3 between 7 and 2007. There are $\frac{2005 - 10}{5} + 1 = 400$ multiples of 5 between 7 and 2007. And there are $\frac{1995 - 15}{15} + 1 = 133$ multiples of 15 between 7 and 2007. Thus, there are a total of $667 + 400 - 133 = 934$ multiples of either or both. So there are $2001 - 934 = 1067$ integers divisible by neither.

Problem 10. What is the sum of the digits of 11^7?

Answer. 38

Solution. By the binomial theorem,

$$11^n = (10+1)^n = \sum_{k=0}^{n} \binom{n}{k} 10^k\,,$$

and so the sum of the digits is *the sum of the digits of* $\displaystyle\sum_{k=0}^{n} \binom{n}{k}$. (Note that the sum of the *digits* of $\sum a_k 10^k$ is the sum of the digits of the a_k's, even if carrying occurs.) Thus, the sum of the digits of 11^7 is

$$\sum_{k=0}^{7} \text{digits of} \binom{7}{k} = 2 \cdot (\text{sum of the digits of } 1, 7, 21, 35) = 38\,.$$

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