1785

Sponsored by: UGA Math Department and UGA Math Club

CIPHERING ROUND / 2 MINUTES PER PROBLEM
NOVEMBER 8, 2008

WITH SOLUTIONS

No calculators are allowed on this test. 2 minutes per problem, 10 points for each correct answer.

Problem 1. What is the area of the circle circumscribed about a 2×2 square?

Answer. 2π

Solution. The diameter of the circle must be the diagonal of the square, which is $2\sqrt{2}$. Therefore, the circle has radius $\sqrt{2}$ and area 2π.

Problem 2. An inflatable beach ball has the shape of a regular dodecahedron. If its surface area is doubled, then by what factor does its volume increase?

Answer. $2^{3/2} = 2\sqrt{2}$

Solution. The surface area varies as the square of the linear dimensions and the volume varies as the cube. So the volume will increase by a factor of $2^{3/2} = 2\sqrt{2}$.

Problem 3. A car was driven 360 miles at constant speed. If the trip had been taken 5 mph slower, it would have taken an extra hour. What was the speed of the trip, in mph?

Answer. 45

Solution. Let v be the speed of the trip, in mph. Then we have

$$\frac{360}{v} + 1 = \frac{360}{v-5},$$

so $v(v-5) = 360 \cdot 5$. Solving, we find $v^2 - 5v - 1800 = (v+40)(v-45) = 0$, so $v = 45$.

Problem 4.

$$(i+1)^{10} + (i-1)^{10} = ?$$

Answer. 0

Solution. The two terms are complex conjugates of one another, so this is twice the real part of $(i+1)^{10}$. Since $i+1$ has equal real and imaginary parts (i.e., lies on the $45°$ ray from the origin), raising it to the tenth power will result in a complex number on the positive imaginary axis.

Problem 5. Express in terms of the fewest number of square roots possible:

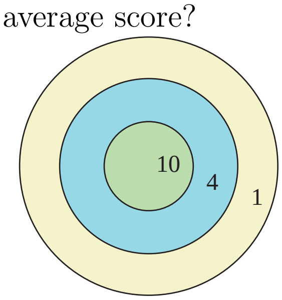
$$\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$$

Answer. 4

Solution. Let $x=\sqrt{7+4\sqrt{3}}+\sqrt{7-4\sqrt{3}}$. Then $x^2=(7+4\sqrt{3})+(7-4\sqrt{3})+2\sqrt{(7+4\sqrt{3})(7-4\sqrt{3})}=14+2\sqrt{49-48}=16$. Since $x>0$, we must have $x=4$.

A more interesting observation is that $\sqrt{7+4\sqrt{3}}=2+\sqrt{3}$ and $\sqrt{7-4\sqrt{3}}=2-\sqrt{3}$, so their sum is 4. When can we express numbers of the form $\sqrt{a\pm\sqrt{b}}$ (a and b rational numbers) as $\sqrt{c}\pm\sqrt{d}$ (for appropriate rational numbers c and d)?

Problem 6. A dartboard consists of three concentric circles of radius $2''$, $4''$, and $6''$, and points are assigned as indicated when a dart lands in the various regions. If Zach throws many, many darts at the dartboard, never misses the board and is equally likely to hit any point in the board, what is his long-term average score?



Answer. 3

Solution. Since the inner, middle, and outer regions have areas in the ratio $1:3:5$, the expected score on one throw will be

$$\frac{1}{9}(1\cdot 10+3\cdot 4+5\cdot 1)=\frac{27}{9}=3\,.$$

Zach's expected score (i.e., average score in the long run) will be 3.

Problem 7. What is the largest value of the function

$$f(x)=2\cos x-\cos 2x\,?$$

Answer. 3/2

Solution. This can be done with calculus, but here is a completely elementary solution. Since $\cos 2x = 2\cos^2 x - 1$,

$$f(x) = 2\cos x - \cos 2x = -2\cos^2 x + 2\cos x + 1 = -2\big(\cos^2 x - \cos x - \tfrac{1}{2}\big)$$
$$= -2\big((\cos x - \tfrac{1}{2})^2 - \tfrac{3}{4}\big)\,.$$

From this we see that $f(x) \le (-2)(-\dfrac{3}{4}) = \dfrac{3}{2}$.

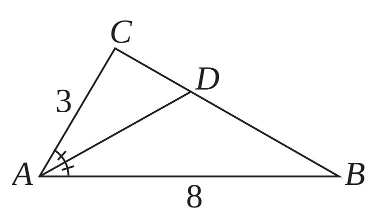
Problem 8. Box A contains 2 red marbles and 1 black marble. Box B contains 3 red marbles and 2 green marbles. Stephanie selects a box at random and then chooses a marble from it at random. If she picks a red marble, what is the probability that she selected Box B?

Answer. $9/19 \approx 0.474$

Solution. The probability of selecting Box A and a red marble from it is $\dfrac{1}{2}\cdot\dfrac{2}{3}$. The probability of selecting Box B and a red marble from it is $\dfrac{1}{2}\cdot\dfrac{3}{5}$. Thus, the probability that Stephanie selects a red marble is $\dfrac{1}{2}\cdot\dfrac{19}{15}$. Thus, the probability that she selected Box B, *given* that she picked a red marble, is

$$\frac{\frac{1}{2}\cdot\frac{3}{5}}{\frac{1}{2}\cdot\frac{19}{15}} = \frac{\frac{3}{5}}{\frac{19}{15}} = \frac{9}{19}\,.$$

Problem 9. In triangle $\triangle ABC$, $AB = 8$, $AC = 3$, $\angle BAC = 60°$. If $\overline{AD}$ bisects $\angle BAC$, then what is CD?



Answer. 21/11

Solution. By the law of cosines,

$$BC = \sqrt{3^2 + 8^2 - 2\cdot 3\cdot 8\cdot\cos(60°)} = \sqrt{9 + 64 - 24} = 7\,.$$

Since an angle bisector divides the opposite side in the same ratio as the adjacent sides, we have $CD/DB = 3/8$, and so $CD = \dfrac{3}{11}BC = \dfrac{3}{11}\cdot 7 = \dfrac{21}{11}$.

Problem 10. What is the smallest positive integer N whose *reciprocal* has a decimal expansion that repeats after every 4 digits and no sooner? (For example, $1/7 = 0.\overline{142857}\ldots$ has a decimal expansion that repeats after every 6 digits.)

Answer. 101

Solution. If the decimal expansion repeats every 4 digits, then

$$10^4 \cdot \frac{1}{N} - \frac{1}{N} = \frac{1}{N}(10^4 - 1)$$

must be an integer. That is, N must be a divisor of $9999 = 9 \cdot 1111 = 9 \cdot 11 \cdot 101$, and $1/3$, $1/9$, $1/11$, $1/33$, and $1/99$ all repeat after one or two digits. The smallest positive integer is $N = 101$.

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