1785

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 20, 2012

**WITH SOLUTIONS**

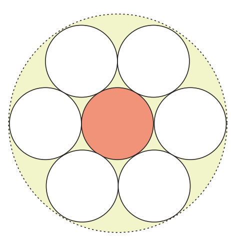
**No calculators are allowed on this test.** 2 minutes per problem, 10 points for each correct answer.

**Problem 1.** A man can saw a long cylindrical log into 5 cylindrical pieces in 5 minutes. Working at exactly the same pace, into how many such pieces can he saw it in 10 minutes?

**Answer.** 9

**Solution.** He can make 4 cuts through the log in 5 minutes, hence 8 cuts in 10 minutes. (No snoring allowed.)

**Problem 2.** Six circles of radius 1 are packed tightly between a circle of radius 1 and an outer circle, as pictured. What portion of the area of the outer ring is covered by the six circles?



**Answer.** 3/4

**Solution.** The outer circle has radius 3, so the area of the outer ring is $8\pi$, of which $6\pi$ is covered by the six circles.

**Problem 3.** Solve for $x$:

$$2\log_3 x - \log_3(2x+7) = 2\,.$$

**Answer.** 21

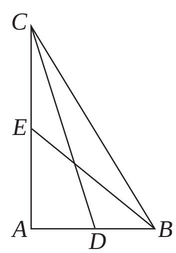
**Solution.** We have $\log_3\left(\frac{x^2}{2x+7}\right) = 2$, so $x^2 = 9(2x+7)$. That is, $x^2 - 18x - 63 = (x-21)(x+3) = 0$. Thus, $x = 21$ or $x = -3$, but we cannot take the log of a negative number, so we discard the second root.

**Problem 4.** A drawer contains 6 red socks and 4 black socks. Two socks are selected randomly (without replacement). What is the probability that the socks are the same color?

**Answer.** $7/15 \approx 46.7\%$

**Solution.** With probability $6/10$ we first pick a red sock. The probability of picking another red sock at that point is $5/9$, so the probability of success with a pair of red socks is $1/3$. With probability $4/10$ we first pick a black sock. The probability of picking another black sock at that point is $3/9$, so the probability of success with a pair of black socks is $2/15$. The probability of success, then, is $\frac{1}{3} + \frac{2}{15} = \frac{7}{15}$.

**Problem 5.** $\overline{BC}$ is the hypotenuse of right triangle $\triangle ABC$. $D$ is the midpoint of $\overline{AB}$ and $E$ is the midpoint of $\overline{AC}$. If $CD = 7$ and $BE = 4$, then what is $BC$?




**Answer.** $2\sqrt{13}$

**Solution.** Let $AD = x$ and $AE = y$. Then $x^2 + (2y)^2 = 49$ and $(2x)^2 + y^2 = 16$. Adding these, we get $5(x^2+y^2) = 65$, so $x^2+y^2 = 13$. Therefore, $BC = 2\sqrt{x^2+y^2} = 2\sqrt{13}$.

**Problem 6.** The polynomial $f(x) = (-4x+3)^7$ is written out as a sum of multiples of powers of $x$. Find the sum of the coefficients.

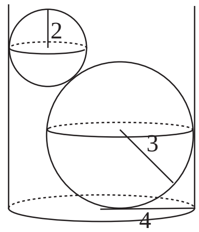
**Answer.** $-1$

**Solution.** By the binomial theorem,

$$f(x) = \sum_{k=0}^{7} \binom{7}{k} (-4x)^k 3^{7-k} = \sum_{k=0}^{7} \binom{7}{k} (-4)^k 3^{7-k} x^k \,,$$

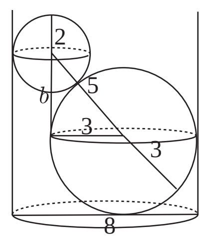
and so we want $\sum_{k=0}^{7} \binom{7}{k}(-4)^k 3^{7-k} = (-4+3)^7 = (-1)^7 = -1$. Of course, this was silly: The sum of the coefficients of any polynomial is obtained by evaluating at $x = 1$, so $f(1) = (-1)^7 = -1$.

**Problem 7.** A ball of radius 3 is put at the bottom of a cylindrical can of radius 4, touching the side of the can. We then put a ball of radius 2 on top of it, so that it is tangent to the opposite side. How high above the bottom of the can will the top of the second ball be?




**Answer.** 9

**Solution.** Since the diameter of the can (8) is the sum of the diameter of the bottom ball (6) and the radius of the top ball (2), the center of the top ball is directly above the leftmost point of the bottom ball. Thus, we have $b = 4$, and the vertical distance from the bottom of the can to the top of the second ball is $3 + 4 + 2 = 9$.




**Problem 8.** How many 3-digit numbers with one each of the digits 5, 7, and 9 are divisible by 11?

**Answer.** 2

**Solution.** The tens digit and the sum of the ones and hundreds digits must differ by a multiple of 11. But they all sum to 21. The only option is $21 = 5 + 16$. (More concretely, if the 3-digit numeral is $abc$, then we have $a+b+c = 21$ and $a-b+c = 11$, so, $2b = 10$ and $b = 5$. Note that $a-b+c = 0$ is impossible because 21 is odd.) Thus, 5 must be the middle digit, and we have either 759 or 957.

**Problem 9.** Three point masses with masses 1, 2, and 2 are placed, equally spaced, on the unit circle. How far from the center of the circle is the center of mass?

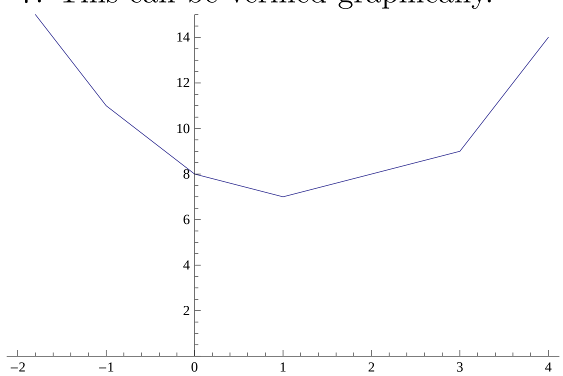
**Answer.** 1/5

**Solution.** Putting masses with values 1 at each of the three vertices of the equilateral triangle balances at the center. Effectively, we have a mass of 3 at the center and masses with value 1 at two vertices. The center of mass of the latter two is at the midpoint of the interval joining them, which is distance 1/2 from the center (e.g., $\cos 60° = 1/2$). So we put a mass of 2 at distance 1/2 and a mass of 3 at the center, and the center of mass is at distance $\frac{2}{5} \cdot \frac{1}{2} = \frac{1}{5}$ from the center.

**Problem 10.** Find the smallest value of $f(x) = |x+1| + |x| + |x-1| + 2|x-3|$.

**Answer.** 7

**Solution.** The minimum occurs at the *median* of the "data set" $-1$, 0, 1, 3, 3, which is $x = 1$. Then $f(1) = 7$. This can be verified graphically.




The function is piecewise-linear, with changes in slope only at the given "data points." To the left of $x = 1$ we have three segments of negative slope, and to the right of $x = 1$ we have two segments of positive slope, so $x = 1$ is the minimum point.

**Authors.** Written by Ted Shifrin with help from Mo Hendon.