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CIPHERING ROUND / 2 MINUTES PER PROBLEM
NOVEMBER 16, 2013

WITH SOLUTIONS

Problem 1. Ted and Valery's ages add up to 35. If Ted is twice as old as Valery was 5 years ago, how old is Ted?

Answer. 20 (years old)

Solution. Let T and V be Ted and Valery's ages. Then

$$T + V = 35, \quad T = 2(V - 5).$$

Solve carefully and you'll find $T = 20$ and $V = 15$.

Problem 2. Suppose you know $|x - 2| \leq 1$. What is the largest possible value of $|x^2 - 4|$?

Answer. 5

Solution. If $|x - 2| \leq 1$, then $-1 \leq x - 2 \leq 1$, so $1 \leq x \leq 3$. Squaring, we get $1 \leq x^2 \leq 9$, and so $-3 \leq x^2 - 4 \leq 5$. In particular, $|x^2 - 4| \leq 5$, with equality when $x = 3$.

Problem 3. Find all real values of x that satisfy the equation

$$e^x + 1 = 12e^{-x}$$

Answer. $\ln(3)$

Solution. Rewrite the equation by multiplying by e^x :

$$(e^x)^2 + e^x = 12.$$

Then treat the resulting equation as a quadratic in $u = e^x$:

$$u^2 + u - 12 = 0.$$

This has roots $u = 3$ and $u = -4$. $e^x \neq -4$ for real x , and so the only solution comes from $e^x = 3$: $x = \ln(3)$.

Problem 4. Paul took 1 minute to solve the first problem on an 8 question math test, and each successive problem took twice as long as the preceding problem. Assuming he took a 1 minute break between problems, how long did it take Paul to complete the test?

Answer. 262 (minutes)

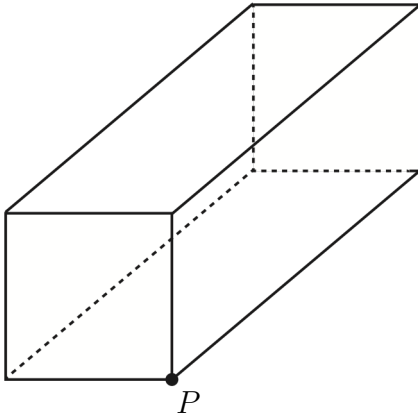
Solution. Paul's times on the 8 questions form a geometric sequence $1, 2, 2^2, \dots, 2^7$. These sum to $\frac{2^8-1}{2-1} = 255$. Add in 7 minutes for breaks, and Paul took a total of 262 minutes.

Problem 5. Find the smallest integer k greater than 1 such that k divided by i has remainder 1 for all i in the set $\{2, 3, 4, 5, 6, 7, 8\}$.

Answer. 841

Solution. If k is the smallest such integer, then $k - 1$ is the smallest integer divisible by 2, 3, 4, 5, 6, 7, and 8, i.e., their least common multiple. The LCM is $3 \cdot 5 \cdot 7 \cdot 8 = 840$, so $k = 841$.

Problem 6. We define the distance between two points on the surface of a $1 \times 1 \times 2$ rectangular box to be the length of the shortest path (on the surface of the box) which joins them. With this definition, the circle of radius 1 centered at P is the set of all points which are a distance of 1 from P . What is the circumference of this circle?



Answer. $3\pi/2$

Solution. The circle consists of three circular arcs, one on each of the rectangular faces meeting at P . Each of the arcs is a quarter circle, so each has length $\pi/2$. So the circumference is $3\pi/2$.

Problem 7. Find the sum of the solutions of

$$x^4 - 76x^3 + 962x^2 - 2900x + 2013 = 0$$

Answer. 76

Solution. If a polynomial $p(x)$ factors as

$$p(x) = (x - a)(x - b)(x - c)(x - d),$$

then

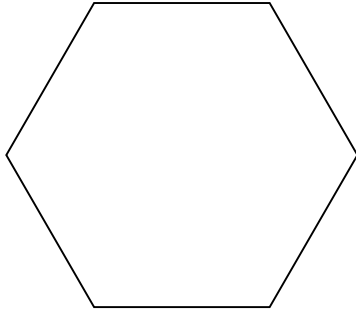
$$p(x) = x^4 - (a + b + c + d)x^3 + \cdots + abcd.$$

So the sum (or product) of the roots can be seen in the coefficients. In this case, the sum of the roots is 76.

In fact, this polynomial factors as

$$(x - 1)(x - 3)(x - 11)(x - 61).$$

Problem 8. If three vertices of a regular hexagon are chosen at random, what is the probability that they are the vertices of a right triangle?



Answer. $3/5$ or 60%

Solution. Three vertices form a right triangle if and only if they include two antipodal points. There are 3 pairs of antipodal points, and any of these can be paired with 4 other points, so there are 12 right triangles. There are $\binom{6}{3} = 20$ total triangles. So the probability is $\frac{12}{20} = \frac{6}{10} = \frac{3}{5} = 60\%$.

Problem 9. How many degree 4 monomials are there in the variables w, x, y, z ? A degree 4 monomial is a term $w^a x^b y^c z^d$ such that a, b, c and d are integers with $0 \leq a, b, c, d \leq 4$ and $a + b + c + d = 4$.

Answer. 35

Solution. We need to partition 4 into a sum of exactly 4 nonnegative integers. Here's one way to think of this: Start with 7 "placeholders", 4 for the integers and 3 for the plus signs:

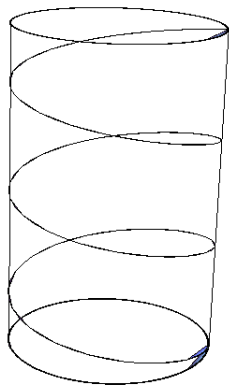
_____ .

Any choice of placement for the 3 + signs determines a partition; for example,

_____ + _____ + _____ .

corresponds to $1 + 1 + 0 + 2$, which in turn corresponds to the monomial wxz^2 . So there are $\binom{7}{3} = 35$ such monomials.

Problem 10. A string will wrap around the base of a certain cylinder exactly 5 times. If instead the same string spirals tightly to the top, it goes around the cylinder exactly 3 times. If the radius of the cylinder is 1, what is the height of the cylinder? Your answer should be written as an integral multiple of π .



Answer. 8π

Solution. If you “unwrap” the cylinder 3 times, you’ll see a rectangle whose base is 3 times the circumference of the cylinder — $3(2\pi r) = 6\pi$ — and whose height h is the unknown we seek. Since the string would wrap 5 times around the base, its length is $5(2\pi r) = 10\pi$. This is a diagonal of the unwrapped rectangle, and so $(6\pi)^2 + h^2 = (10\pi)^2$. Thus, $h = 8\pi$.

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