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CIPHERING ROUND / 2 MINUTES PER PROBLEM

OCTOBER 22, 2016

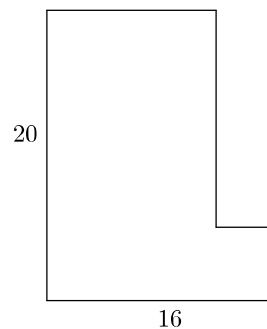
WITH SOLUTIONS

Problem 1. A rectangular prism has edge lengths 6, 8, and 24. What is the length of the interior diagonal?

Answer. 26

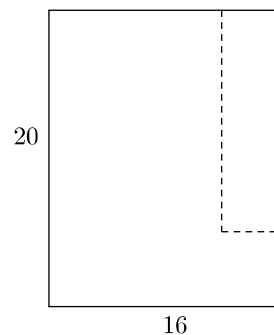
Solution. By the distance formula, the answer is $\sqrt{6^2 + 8^2 + 24^2} = \sqrt{676} = 26$. To simplify the arithmetic, use instead a box with edge lengths 3, 4, and 12, and take advantage of the familiar Pythagorean triples 3, 4, 5 and 5, 12, 13.

Problem 2. What is the perimeter of the polygon? All angles are 90° .

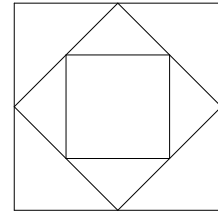


Answer. 72

Solution. The perimeter is the same as this polygon:

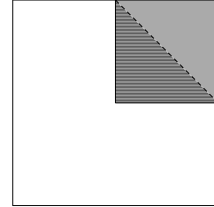


Problem 3. Start with a square. Connect the midpoints of adjacent sides to form a second square. Connect the midpoints of the second square to form a third square. What is the ratio of the area of the first square to the area of the third?

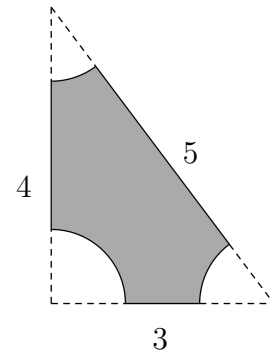


Answer. 4

Solution. In the diagram to the right, the areas of the two shaded triangles are equal. This shows that the area of the n th square is twice the area of the $(n + 1)$ st, and so the area of the n th square is 2^k times the $(n + k)$ th. This question is the $k = 2$ case.



Problem 4. What is the area of the shaded region? The curves are arcs of circles of radius 1, centered at the vertices, and the triangle has side lengths 3, 4, 5.



Answer. $6 - \frac{\pi}{2}$ or $\frac{12 - \pi}{2}$

Solution. The area of the triangle is $\frac{1}{2} \cdot 3 \cdot 4 = 6$. The three circular sectors combine for an angle of 180° , and so their combined area is $\frac{1}{2}\pi \cdot r^2 = \frac{\pi}{2}$. So the total area is $6 - \frac{\pi}{2}$.

Problem 5. Solve for x :

$$e^{2x} - 4e^{x+1} + 4e^2 = 0.$$

Answer. $\ln(2e)$ or $1 + \ln(2)$

Solution. This factors as $(e^x - 2e)^2$, and so $e^x = 2e$, i.e., $x = \ln(2e) = 1 + \ln(2)$.

Problem 6. 10 apples and 5 bananas cost \$45. 5 apples and 10 bananas cost \$30. What is the difference between the price of an apple and the price of a banana?

Answer. \$3 (i.e., 3 dollars)

Solution. Let A be the cost of an apple and B the cost of a banana. The given information shows that

$$\begin{aligned}10A + 5B &= 45, \\5A + 10B &= 30.\end{aligned}$$

Subtract, then divide by 5:

$$\begin{aligned}5A - 5B &= 15, \\A - B &= 3.\end{aligned}$$

Note that we did not need to solve for A and B individually.

Problem 7. If you know that the real number x is closer to 10 than to 15, and closer to 3 than to 2, what is the length of the interval in which x can lie?

Answer. 10

Solution. “Closer to 10 than to 15” simply means $x < 12.5$. Similarly, “closer to 3 than to 2” means $x > 2.5$. So x lies in the interval $(2.5, 12.5)$, which has length 10.

Problem 8. Mo gave a 10 question quiz to 40 students. So far he has graded all of the questions on $\frac{1}{5}$ of the quizzes, and $\frac{1}{5}$ of the questions on the rest of the quizzes. What percentage of the questions has he graded?

Answer. 36%

Solution. You can simply compute the number of questions (out of 400 total) that have been graded:

$$\text{all of the questions on } \frac{1}{5} \text{ of the quizzes: } 10 \cdot \frac{1}{5} \cdot 40 = 80.$$

$$\frac{1}{5} \text{ of the questions on the rest of the quizzes: } \left(\frac{1}{5} \cdot 10\right) \cdot \left(\frac{4}{5} \cdot 40\right) = 64.$$

So he's graded $\frac{144}{400} = 36\%$ of the questions.

Alternatively, you can think in terms of probability. A problem has not been graded if it's not on one of the quizzes that's been completely graded ($p = 4/5$) and it's not in the $1/5$ of the problems on each test that have been graded (also $p = 4/5$). So it's not been graded with probability $(4/5)^2 = 16/25 = 64\%$; i.e., 36% of problems have been graded.

Problem 9. If x, y , and z are positive integers satisfying $2^x + 2^y = 2^z$ and $2^x \cdot 2^y = 2^{z+1}$, then what is z ?

Answer. 3 (or $z = 3$)

Solution. We observe first that $2^x + 2^y = 2^z$ implies that $x = y$. For example, if $x > y$, then $2^x + 2^y = 2^y(2^{x-y} + 1)$, which would make $2^{x-y} + 1$ an odd factor of 2^z . Similarly, we cannot have $x < y$.

Since $x = y$, we have $2^z = 2^x + 2^x = 2^{x+1}$, so that $z = x + 1$. Substituting $x = y = z - 1$ into the second given equation yields $2^{2z-2} = 2^{z+1}$, and hence $2z - 2 = z + 1$, so that $z = 3$.

Problem 10. What is the smallest positive integer that has exactly 10 factors? (Hint: It is not 6, which has 4 factors: 1, 2, 3, and 6.)

Answer. 48

Solution. A reasonable first guess is 2^9 , which has the 10 divisors $2^0, 2^1, \dots, 2^9$. But $2^4 \cdot 3$ is smaller and has the 10 divisors $2^0, 2^1, 2^2, \dots, 2^4, 2^0 \cdot 3, 2^1 \cdot 3, \dots, 2^4 \cdot 3$. To see that $2^4 \cdot 3$ is the smallest possible, recall that if p_1, \dots, p_k are distinct primes and e_1, \dots, e_k are positive integers, then the number of divisors of n is $(e_1 + 1) \cdots (e_k + 1)$. So a number with 10 divisors is either of the form p^9 for a prime p , or of the form $p^4 q$ for distinct primes p, q . The smallest solution of the first kind is clearly 2^9 , while the smallest solution of the second kind is $2^4 \cdot 3^1 = 48$.

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