1785

Sponsored by: UGA Math Department and UGA Math Club

Ciphering Round / 2 minutes per problem
October 21, 2017

WITH SOLUTIONS

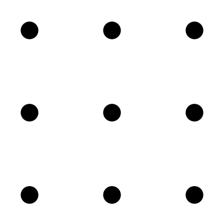
Problem 1. In a recent football game, team A had three times as many points as team B. B then scored another touchdown (7 points), after which A had twice as many points as B. What was the combined score of the two teams after that touchdown?

Answer. 63

Solution. Let a and b be the scores of A and B before the touchdown. Then $a = 3b$ and $a = 2(b + 7)$. Solving, $a = 42$ and $b = 14$, so the combined score after the touchdown is $42 + (14 + 7) = 63$.

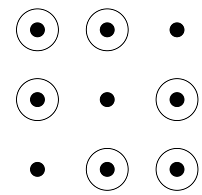
Note. This *almost* happened in a recent UGA-Samford football game: Georgia had a 42-14 lead ("team A had three times as many points. . ."), with Samford threatening to score late in the 4th quarter, but a stout defensive stand maintained a final score of 42-14.

Problem 2. What is the maximum number of dots you can choose from a 3 by 3 grid of dots, if no 3 of the chosen dots can be in the same row, the same column, or the same diagonal?



Answer. 6

Solution. Since you can choose no more than two dots from each row, it is not possible to choose more than six dots. That six is possible is shown in the right-hand figure.



More generally, one can ask for the maximum number of dots that can be selected from an $n \times n$ grid with no three lying on the same line. Call this $f(n)$. Clearly, $f(n) \leq 2n$, and computer calculations have shown that $f(n) = 2n$ for all $n \leq 46$. It is conjectured that this pattern eventually breaks down, and that $f(n) < 2n$ for all large enough n. In fact, a heuristic argument of Richard Guy and Patrick Kelly, as corrected by Ed Pegg, suggests (but does **not** prove) that for any constant $c > \pi/\sqrt{3}$, one has $f(n) < cn$ for all sufficiently large n. Note that $\pi/\sqrt{3} \approx 1.814$. In the opposite direction, it has been proved that for any $c' < 3/2$, one has $f(n) > c'n$ for all large enough n.

See: R.K. Guy and P.A. Kelly. *The no-three-in-line problem.* Canadian Mathematical Bulletin 11 (1968), 527–531.

A. Flammenkamp. *The No-Three-in-Line Problem.* `http://wwwhomes.uni-bielefeld.de/achim/cgi/no3in/readme.html`

Problem 3. Find $\displaystyle\sum_{p=1}^{\infty}\left(\sum_{q=1}^{\infty}\left(\frac{1}{3}\right)^{q}\right)^{p}$.

Answer. 1

Solution. First, recall that if $|r| < 1$, then $\sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$. Hence, the inner sum is

$$\sum_{q=1}^{\infty}\left(\frac{1}{3}\right)^{q} = \frac{1/3}{1-1/3} = 1/2,$$

and so the outer sum is

$$\sum_{p=1}^{\infty}\left(\frac{1}{2}\right)^{p} = \frac{1/2}{1-1/2} = 1.$$

Problem 4. How many 1s are in the base 2 expansion of the number whose base 8 expansion is 2017?

Answer. 5

Solution. Notice that

$$\begin{aligned}(2017)_8 &= 2\times 8^3 + 0\times 8^2 + 1\times 8^1 + 7\times 8^0\\ &= 2^{10} + 2^3 + 2^2 + 2 + 1\\ &= (10000001111)_2.\end{aligned}$$

You can simply write each base 8 digit as a three-digit base 2 number and concatenate the results.

Problem 5. How many ordered pairs of positive integers (x, y) satisfy $x + y \leq 100$?

Answer. 4950

Solution. If $x = 1$, there are 99 solutions: $(1, 1), \ldots, (1, 99)$.
If $x = 2$, there are 98 solutions: $(2, 1), \ldots, (2, 98)$.

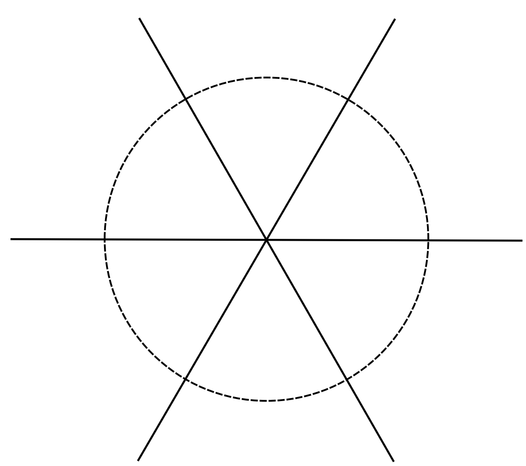
$\vdots$

If $x = 99$, there is one solution: $(99, 1)$. Thus, the total number of solutions is $\sum_{k=1}^{99} k = \frac{99\cdot 100}{2} = 4950$.

Problem 6. What is the largest number of lines you can draw through $(0, 0)$ in the xy-plane with the property that the angle between any two of them is the same?

Answer. 3

Solution.



Notice that a circle centered at the origin will meet these lines in the vertices of a regular polygon, and a polygon with more than 6 such sides (3 lines) is clearly not equiangular.

Problem 7. How many of the coefficients of $(2x+\frac{1}{2}y)^8$ are integers, after simplifying?

Answer. 6

Solution. Clearly the coefficients of $x^8, x^7y, x^6y^2, x^5y^3$, and x^4y^4 are integers, while the coefficient of y^8 is not, so we can focus on the other terms in the expansion:

$$\cdots + \binom{8}{3}(2x)^3\left(\frac{1}{2}y\right)^5 + \binom{8}{2}(2x)^2\left(\frac{1}{2}y\right)^6 + \binom{8}{1}(2x)^1\left(\frac{1}{2}y\right)^7 + \cdots.$$

We have

$$\binom{8}{3} = \frac{8\cdot 7\cdot 6}{3\cdot 2\cdot 1} = 8\cdot 7, \quad \text{so the coefficient of } x^3y^5 \text{ is an integer,}$$

$$\binom{8}{2} = \frac{8\cdot 7}{2} = 4\cdot 7, \quad \text{so the coeff. of } x^2y^6 \text{ is } 4\cdot 7\cdot 2^2\cdot\left(\frac{1}{2}\right)^6, \textbf{ not} \text{ an integer,}$$

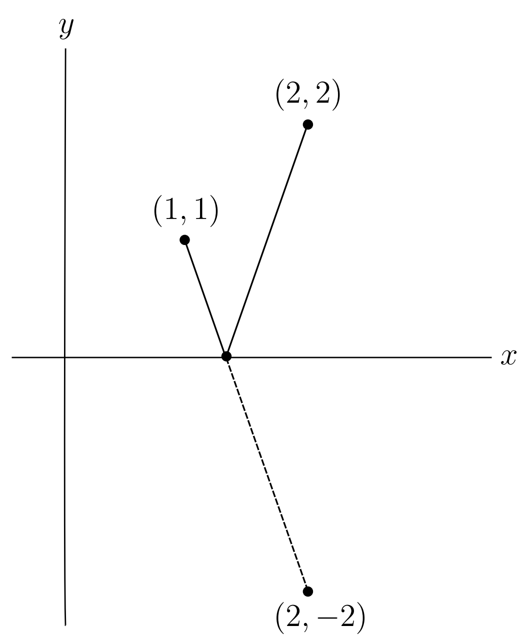
$$\binom{8}{1} = 8, \quad \text{so the coeff. of } xy^7 \text{ is } 8\cdot 2\cdot\left(\frac{1}{2}\right)^7, \textbf{ not} \text{ an integer.}$$

So three of the coefficients are not integers, and the remaining six are integers.

Problem 8. What is the length of the shortest path in the xy-plane that starts at $(1,1)$, touches the x-axis, and ends at $(2,2)$?

Answer. $\sqrt{10}$

Solution. The shortest path will be the reflection of the shortest path from $(1,1)$ to $(2,-2)$, which has length $\sqrt{10}$.



Problem 9. If x and y are positive integers satisfying $\ln(x+y) = \ln(x)+\ln(y)$, what is x^2+y^2?

Answer. 8

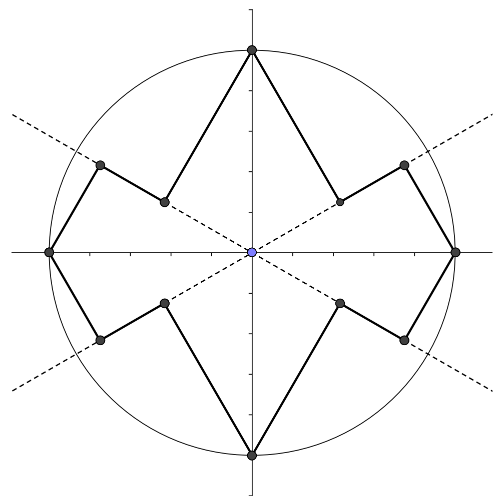
Solution. Exponentiating both sides yields $x+y = e^{\ln(x+y)} = e^{\ln(x)+\ln(y)} = xy$, so that

$$y = \frac{x}{x-1} = 1 + \frac{1}{x-1}.$$

This has $(x, y) = (2, 2)$ as its unique positive integer solution, so that $x^2 + y^2 = 2^2 + 2^2 = 8$.

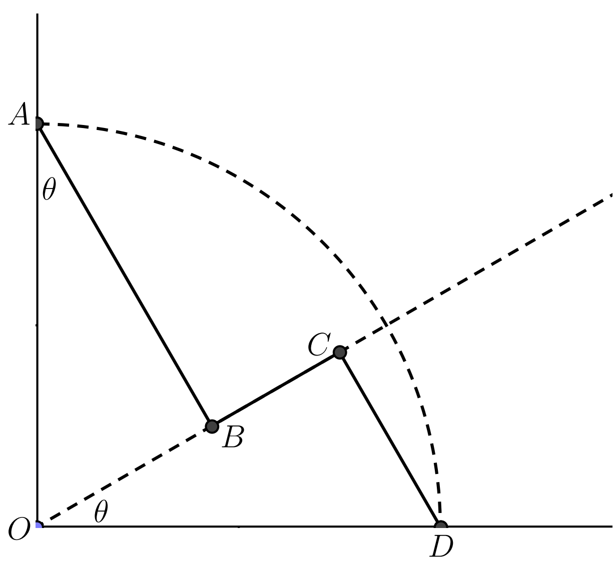
Problem 10. Start with a circle of radius 1 centered at $(0, 0)$. Draw two lines making an angle of $30°$ with the x-axis. Drop perpendiculars from these lines to the points $(\pm 1, 0)$ and $(0, \pm 1)$. Join the perpendiculars along the $30°$ lines to form a closed polygon.

What is the perimeter of this polygon?



Answer. $4\sqrt{3}$

Solution. Label the points O, A, B, C, and D as in the following diagram.



Observe that ΔOBA is congruent to ΔDCO, and that $AB = \cos\theta$, $BC = \cos\theta - \sin\theta$, and $CD = \sin\theta$. Thus, the contribution to the perimeter from the first quadrant is

$$\cos\theta + (\cos\theta - \sin\theta) + \sin\theta = 2\cos\theta,$$

and the entire perimeter is $4 \cdot 2\cos\theta = 8\cos\theta = 8 \cdot \frac{\sqrt{3}}{2} = 4\sqrt{3}$.

Authors. Written by Mo Hendon, Paul Pollack, and Peter Woolfitt.