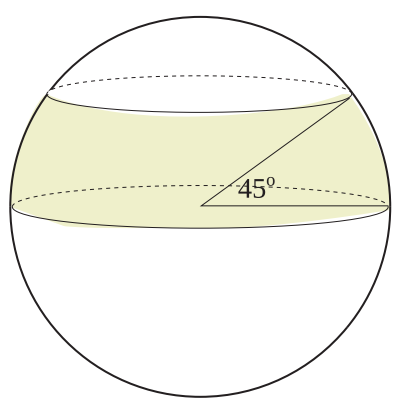
Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 200 points.

Problem 1.(The globe) What portion of the earth's surface is located between the equator and the 45^0 latitude?



Answer.

$$\frac{\sqrt{2}}{4} \approx .3535 \approx 35.35\%$$

(Will accept $.35 \pm .2$ and $35\% \pm 2\%$ if a numerical (not exact) answer was given. Example: 33% is accepted, 1/3 is not. Will also accept $\sin(45^0)/2$ or $\cos(45^0)/2$.)

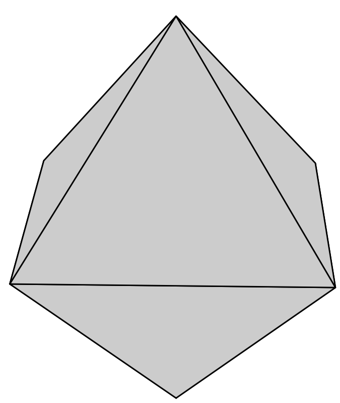
Solution. Project our area to the vertical axis: you get the interval $[0, \sin 45^0] = [0, \sqrt{2}/2]$. A theorem of Archimedes says that the area is proportional to the length of this interval. (Prove it; it is fun!)

Therefore, the portion of our surface area is

$$\frac{\sqrt{2}/2}{2} = \frac{\sqrt{2}}{4}$$

Remark: in symplectic geometry, there exist very interesting and very deep generalizations of Archimedes' theorem and the projection to the vertical axis called "Duistermaat-Heckman measure" and "the moment map".

Problem 2.(Planets) 6 planets of radius 1 are centered at the vertices of a regular octahedron of side length 3. A point on the surface of one of the planets is called *invisible* if it cannot be seen from some other planet. What is the total area of the set of invisible points?



Answer. 4π.

Solution. Let us fix an origin in our system. Now let us pick an arbitrary direction and call it the "northern direction".

Each planet has a "north pole". If the direction was chosen randomly enough (mathematically speaking, "outside of a set of measure zero") then there is now exactly one "north-most" planet. Then its north pole is invisible and the north poles of the all other planets are visible!

This shows that the invisible points can be associated with directions, i.e. with points on a unit sphere. Since the planets have radius 1 as well, the invisible area equals the area of the unit sphere, i.e. $4\pi \cdot 1^2 = 4\pi$.

Note that this proof works for *any number of planets* (and even in arbitrary dimension!)

Problem 3.(Pascal's triangle) What are the first two digits after the decimal point in

$$(\sqrt{3}+\sqrt{2})^{2004}?$$

(Hint: Compare with $(\sqrt{3}-\sqrt{2})^{2004}$.)

Answer. 99

Solution. We claim that

$$(\sqrt{3}+\sqrt{2})^{2004} + (\sqrt{3}-\sqrt{2})^{2004}$$

is an integer. Indeed, using the binomial formula,

$$\begin{aligned}(\sqrt{3}+\sqrt{2})^{2004} &= \sqrt{3}^{2004} + \binom{2004}{1}\sqrt{3}^{2003}\sqrt{2} + \binom{2004}{2}\sqrt{3}^{2002}\sqrt{2}^{2} + \dots \\ &= 3^{1002} + \binom{2004}{1}3^{1001}\sqrt{6} + \binom{2004}{2}3^{1001}2 + \binom{2004}{3}3^{1000}2\sqrt{6} + \dots\end{aligned}$$

and

$$\begin{aligned}(\sqrt{3}-\sqrt{2})^{2004} &= \sqrt{3}^{2004} - \binom{2004}{1}\sqrt{3}^{2003}\sqrt{2} + \binom{2004}{2}\sqrt{3}^{2002}\sqrt{2}^{2} - \dots \\ &= 3^{1002} - \binom{2004}{1}3^{1001}\sqrt{6} + \binom{2004}{2}3^{1001}2 - \binom{2004}{3}3^{1000}2\sqrt{6} + \dots\end{aligned}$$

Therefore, in the sum

$$(\sqrt{3}+\sqrt{2})^{2004}+(\sqrt{3}-\sqrt{2})^{2004}$$

all terms with $\sqrt{6}$ cancel out, and the result is an integer. On the other hand, $\sqrt{3}-\sqrt{2}\approx .3$, so $(\sqrt{3}-\sqrt{2})^{2004}$ is a very small (and positive!) number. Hence, $(\sqrt{3}+\sqrt{2})^{2004}$ ends with $.9999\ldots$ (Using a computer, one can check that it ends with 997 9's.)

Problem 4.(Rubik Hypercube) Imagine an n-dimensional 3 by 3 by 3 $\ldots$ by 3 hypercube, consisting of 3^n cells, smaller cubes. How many diagonals does it have? Here, a *diagonal* is defined to be a straight line consisting of 3 distinct cells. Note that you already solved the $n=3$ case of this problem in the main test, and the answer there was $f(3)=49$.

Your answer should be a concise formula for the function $f(n)$ expressing the number of diagonals as a function of dimension n. The answer will be graded on elegance, in addition to correctness, of course.

Answer.

$$\frac{5^n-3^n}{2}$$

Will also accept the answer

$$\sum_{i=1}^{n}\binom{n}{i}3^i2^{n-i}=\sum_{i=0}^{n-1}\binom{n}{i}2^i3^{n-i}$$

or equivalent, but award only 40 points out of 50.

Solution. Fix a diagonal (c_1,c_2,c_3) and look at the first coordinate of the cells c_1,c_2,c_3. These coordinates are either constant (and then there are 3 choices), or increasing, or decreasing. So, there are 5 possibilities of what can happen. Repeating this with the second, third etc. coordinates, we see that there are 5^n possibilities.

Of these, 3^n will correspond to the cases where every coordinate is constant (i.e. $c_1=c_2=c_3$), so these will not correspond to 3 *distinct* cells, i.e. to real diagonals. Finally, we have to divide by 2 since each diagonal has 2 possible directions.