Sponsored by: UGA Math Department and UGA Math Club

Team Round / 45 min / 150 points

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 150 points.

For problem 3, the answer should be an exact expression, such as $\pi/2$, $\sqrt{3} + 1$, $8/3$, etc. No approximate answers will be accepted.

Problem 1. (Five secret numbers) Suppose there are 5 numbers whose pairwise sums are

$5, 9, 20, 24, 31, 35, 39, 42, 46, 61$

What are the original 5 numbers? Write them in increasing order.

Answer.

$-3, 8, 12, 27, 34$

Solution. Denote the numbers $a \leq b \leq c \leq d \leq e$. Then

$a + b = 5$, $d + e = 61$, and

$a + b + c + d + e = \frac{5 + 9 + 20 + 24 + 31 + 35 + 39 + 42 + 46 + 61}{4} = \frac{312}{4} = 78$
Therefore,
\[ c = (a + b + c + d + e) - (a + b) - (d + e) = 78 - 5 - 61 = 12 \]
The next largest number after \( a + b \) is \( a + c = 9 \), so \( a = 9 - 12 = -3 \). Then \( b = (a + b) - a = 5 - (-3) = 8 \). Similarly, \( e = 34 \) and \( d = 27 \).

**Problem 2. (The last man standing)** \( n \) people stand in a circle. Then, every second person is excluded until only one is left. For example, with 10 people, the order of exclusion is as follows:
\[ 2, 4, 6, 8, 10, 3, 7, 1, 9 \]
so the last remaining person is number 5.

Now start with 2005 people. Who will be the last person standing?

**Answer.** 1963

**Solution.** Denote the answer for \( n \) by \( J(n) \). Then we have:
\[ J(2n) = 2J(n) - 1 \quad \text{and} \quad J(2n + 1) = 2J(n) + 1 \]
Indeed, if there are originally \( 2n \) people, the first \( n \) people to be eliminated are numbered 2, 4, 6, 8, \ldots, 2n. The remaining \( n \) people, numbered 1, 3, 5, \ldots, 2n – 1 will essentially be eliminated exactly as if they were numbered 1, 2, 3, \ldots, \( n \). Thus, \( J(2n) = 2J(n) - 1 \); the argument for the other case is similar.
To compute $J(2005)$, read down the first column and up the second:

$$
egin{align*}
2005 &= 2 \cdot 1002 + 1 \\
1002 &= 2 \cdot 501 \\
501 &= 2 \cdot 250 + 1 \\
250 &= 2 \cdot 125 \\
125 &= 2 \cdot 62 + 1 \\
62 &= 2 \cdot 31 \\
31 &= 2 \cdot 15 + 1 \\
15 &= 2 \cdot 7 + 1 \\
7 &= 2 \cdot 3 + 1 \\
3 &= 2 \cdot 1 + 1 \\
J(2005) &= 2 \cdot 981 + 1 = 1963 \\
J(1002) &= 2 \cdot 491 - 1 = 981 \\
J(501) &= 2 \cdot 245 + 1 = 491 \\
J(250) &= 2 \cdot 123 - 1 = 245 \\
J(125) &= 2 \cdot 61 + 1 = 123 \\
J(62) &= 2 \cdot 31 - 1 = 61 \\
J(31) &= 2 \cdot 15 + 1 = 31 \\
J(15) &= 2 \cdot 7 + 1 = 15 \\
J(7) &= 2 \cdot 3 + 1 = 7 \\
J(3) &= 2 \cdot 1 + 1 = 3 \\
J(1) &= 1
\end{align*}
$$

It can also be shown that if $n = (1b_{m-1}b_{m-2} \cdots b_1b_0)_2$ written in binary, then $J(n) = (b_{m-1}b_{m-2} \cdots b_1b_01)_2$.

**Problem 3. (Two triangles)** In a triangle $ABC$, vertices are connected to the points $A'$, $B'$, $C'$ which divide the corresponding sides with the ratio 2 to 1, as in the picture, to form a small triangle $KLM$ in the center. What is the ratio of area of $ABC$ to the area of $KLM$? (The answer must be greater than 1.)

Answer. 7
Solution. Put the triangle $ABC$ in the 3-dimensional space in the plane $x + y + z = 1$ so that the vertices are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Then the lines are cut out by planes $2x = y$, $2y = z$ and $2z = x$. Solving for $2x = y$, $2y = z$ and $x + y + z = 1$ gives $(1/7,2/7,4/7)$ and the other two vertices of $KLM$ are obtained by rotating this triple around. Let the side of triangle $ABC$ be $x$ and the side of $KLM$ be $y$. We have $x^2 = 2$. The side $y$ is the length of the vector

$$(2/7,4/7,1/7) - (1/7,2/7,4/7) = (1/7,2/7,−3/7)$$

Its length can be found using the distance formula:

$$y^2 = \left(\frac{1}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{−3}{7}\right)^2 = \frac{1^2 + 2^2 + 3^2}{7^2} = \frac{14}{7^2} = \frac{2}{7}$$

The ratio of the areas is

$$\frac{x^2}{y^2} = \frac{2}{2/7} = 7$$

[The ratio of these areas is independent of the shape of the original triangle: when we act on the plane by a linear transformation, all areas are multiplied by the same constant, the absolute value of its determinant.]

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