University of Georgia High School Math Tournament

1785

2005

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 45 min / 150 points

**WITH SOLUTIONS**

**No calculators are allowed on this test.** You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 150 points.

For problem 3, the answer should be an exact expression, such as $\pi/2$, $\sqrt{3}+1$, $8/3$, etc. No approximate answers will be accepted.

**Problem 1. (Five secret numbers)** Suppose there are 5 numbers whose pairwise sums are

$$5, 9, 20, 24, 31, 35, 39, 42, 46, 61$$

What are the original 5 numbers? Write them in increasing order.

**Answer.**

$$-3, 8, 12, 27, 34$$

**Solution.** Denote the numbers $a \le b \le c \le d \le e$. Then

$a + b = 5, \qquad d + e = 61, \quad$ and

$$a+b+c+d+e = \frac{5+9+20+24+31+35+39+42+46+61}{4} = \frac{312}{4} = 78$$

Therefore,

$$c = (a+b+c+d+e) - (a+b) - (d+e) = 78 - 5 - 61 = 12$$

The next largest number after $a+b$ is $a+c=9$, so $a = 9-12 = -3$. Then $b = (a+b) - a = 5 - (-3) = 8$. Similarly, $e = 34$ and $d = 27$.

**Problem 2. (The last man standing)** $n$ people stand in a circle. Then, every second person is excluded until only one is left. For example, with 10 people, the order of exclusion is as follows:

$$2, 4, 6, 8, 10, 3, 7, 1, 9$$

so the last remaining person is number 5.

Now start with 2005 people. Who will be the last person standing?

**Answer.** 1963

**Solution.** Denote the answer for $n$ by $J(n)$. Then we have:

$$J(2n) = 2J(n) - 1 \quad \text{and} \quad J(2n+1) = 2J(n) + 1$$

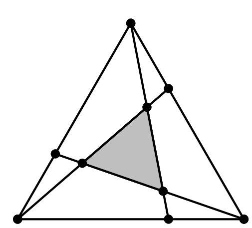
Indeed, if there are originally $2n$ people, the first $n$ people to be eliminated are numbered $2, 4, 6, 8, \ldots, 2n$. The remaining $n$ people, numbered $1, 3, 5, \ldots, 2n-1$ will essentially be eliminated exactly as if they were numbered $1, 2, 3, \ldots, n$. Thus, $J(2n) = 2J(n) - 1$; the argument for the other case is similar.

To compute $J(2005)$, read down the first column and up the second:

$$\begin{array}{rl}
2005 = 2\cdot 1002 + 1 & J(2005) = 2\cdot 981 + 1 = 1963 \\
1002 = 2\cdot 501 & J(1002) = 2\cdot 491 - 1 = 981 \\
501 = 2\cdot 250 + 1 & J(501) = 2\cdot 245 + 1 = 491 \\
250 = 2\cdot 125 & J(250) = 2\cdot 123 - 1 = 245 \\
125 = 2\cdot 62 + 1 & J(125) = 2\cdot 61 + 1 = 123 \\
62 = 2\cdot 31 & J(62) = 2\cdot 31 - 1 = 61 \\
31 = 2\cdot 15 + 1 & J(31) = 2\cdot 15 + 1 = 31 \\
15 = 2\cdot 7 + 1 & J(15) = 2\cdot 7 + 1 = 15 \\
7 = 2\cdot 3 + 1 & J(7) = 2\cdot 3 + 1 = 7 \\
3 = 2\cdot 1 + 1 & J(3) = 2\cdot 1 + 1 = 3 \\
 & J(1) = 1
\end{array}$$

It can also be shown that if $n = (1b_{m-1}b_{m-2}\cdots b_1b_0)_2$ written in binary, then $J(n) = (b_{m-1}b_{m-2}\cdots b_1b_0 1)_2$.

**Problem 3. (Two triangles)** In a triangle $ABC$, vertices are connected to the points $A'$, $B'$, $C'$ which divide the corresponding sides with the ratio 2 to 1, as in the picture, to form a small triangle $KLM$ in the center. What is the ratio of area of $ABC$ to the area of $KLM$? (The answer must be greater than 1.)



**Answer.** 7

**Solution.** Put the triangle $ABC$ in the 3-dimensional space in the plane $x + y + z = 1$ so that the vertices are $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. Then the lines are cut out by planes $2x = y$, $2y = z$ and $2z = x$. Solving for $2x = y$, $2y = z$ and $x + y + z = 1$ gives $(1/7, 2/7, 4/7)$ and the other two vertices of $KLM$ are obtained by rotating this triple around. Let the side of triangle $ABC$ be $x$ and the side of $KLM$ be $y$. We have $x^2 = 2$. The side $y$ is the length of the vector

$$(2/7, 4/7, 1/7) - (1/7, 2/7, 4/7) = (1/7, 2/7, -3/7)$$

Its length can be found using the distance formula:

$$y^2 = \left(\frac{1}{7}\right)^2 + \left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2 = \frac{1^2 + 2^2 + 3^2}{7^2} = \frac{14}{7^2} = \frac{2}{7}$$

The ratio of the areas is

$$\frac{x^2}{y^2} = \frac{2}{2/7} = 7$$

[The ratio of these areas is independent of the shape of the original triangle: when we act on the plane by a linear transformation, all areas are multiplied by the same constant, the absolute value of its determinant.]