Sponsored by: UGA Math Department and UGA Math Club

Team Round / 45 min / 150 points
November 18, 2006

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 50 points, for a total of 150 points.

Problem 1. (Color the cube) In how many ways can one color the six faces of the cube using two red squares, two blue squares and two white squares? (Two colorings are considered to be the same if one can be rotated into the other.)

Answer. 6

Solution. Consider the two red faces. If they are opposite one another, there are two possibilities: the blue squares are either next to each other, or they are not.

Now suppose the red squares are next to each other. We need to choose which of the remaining 4 squares are blue. There are $\binom{4}{2} = 6$ ways but two pairs of them are equivalent: when neither the blue squares nor the white squares are opposite the same-colored square. So there are only 4 new cases. The total is $2 + 4 = 6$.

Second solution. Burnside's theorem is often helpful with such problems, although it is more useful when there are more possibilities.

Problem 2. (Kissing spheres) 4 spheres of radius 1 are placed so that each touches the other three. What is the radius of the smallest sphere that contains all 4 spheres?

Answer.

$$\sqrt{3/2}+1$$

Solution. The centers of the 4 spheres form a regular tetrahedron side 2. The center of the fifth sphere must obviously be at the center of the tetrahedron. An altitude of the tetrahedron will have one end at the centroid of a face, so its length is

$$\sqrt{2^2-((2/3)\sqrt{3})^2}=\sqrt{4-4/3}=2\sqrt{2/3}.$$

So the distance from the center of the fifth sphere to one of the other centers is $(3/4)2\sqrt{2/3}=\sqrt{3/2}$. Hence the radius of the fifth sphere is $\sqrt{3/2}+1$.

There is a more general *Kissing theorem* which says that for five mutually tangent spheres the radii must satisfy the following equation:

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}\right)^2=3\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}+\frac{1}{e^2}\right)$$

Plugging in $a=b=c=d=1$ and solving for e gives $e=-(\sqrt{3/2}+1)$, and the minus sign appears because the last sphere touches the first four "from the other direction".

The *Kissing theorem* actually works in any dimension! In dimension n, take $n+1$ mutually tangent "spheres" and replace the number 3 in the above formula by n.

Problem 3.(Happy squares) In how many ways can one put numbers 1 through 9 in a 3-by-3 square in such a way that numbers in every row are increasing (from left to right) and numbers in every column are increasing (from top to bottom).

Answer. 42

Solution. The number 1 must be in the top-left corner. The number 2 must be immediately to the right or down from it. In each case there are 2 possibilities for the number 3.

On the other hand, 9 must be in the bottom-right corner. The number 8 must be immediately to the left or up from it. In each case there are 2 possibilities for the number 7.

This gives several basic configurations. For each of them we must put in the numbers 4, 5, 6. Enumerating them all (carefully!) gives 42 combinations.

Second solution. This is an immediate application of the *hook formula*, which is a nice fact to learn.

The probability that the square in the top left corner is smaller than the four numbers to the right or down from it is 1/5. For the second square in the first row it is 1/4, etc. Arguing naively, the probability that every square is smaller than the numbers to the right or down should be

$$\frac{1}{5}\left(\frac{1}{4}\right)^2\left(\frac{1}{3}\right)^3\left(\frac{1}{2}\right)^2$$

and the number of good ways is

$$\frac{9!}{5^2\ 4^2\ 3^3\ 2^2} = 42$$

This naive counting can not possibly be rigorous: the events we described are *not* independent; so the probability of all of them happening is not obviously the product of the individual probabilities. But the *hook theorem* says that this naive counting actually works!