Sponsored by: UGA Math Department and UGA Math Club

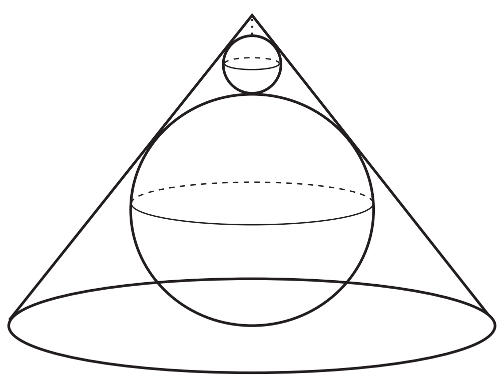
Team Round / 45 min / 210 points
October 13, 2007

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1. (Spheres) An infinite sequence of spheres is inscribed in a right circular cone with base 2, with the largest having radius 1, as shown. Find the volume of that portion of the cone lying outside all the spheres.

Give the answer as a fraction, possibly involving π, $\sqrt{2}$, etc. In other words, give the exact answer, not a decimal approximation.



Answer. $\frac{4\pi}{3} \cdot \frac{104}{63} = \frac{416\pi}{189}$

Solution. Let the height of the cone be h and the slant height a. By similar triangles, we have $\frac{2}{h} = \frac{1}{a-2}$, so $h = 2(a-2)$; since $h^2 + 4 = a^2$, we find that $a = 10/3$ and $h = 8/3$. The radii of the sequence of inscribed spheres gives a geometric sequence; let its ratio be r. Since the sum of all their radii is half the height, we find that $\frac{1}{1-r} = \frac{4}{3}$, and so $r = 1/4$. Thus, the total volume of all the spheres is

$$\frac{4}{3}\pi \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^{3n} = \frac{4}{3}\pi \frac{1}{1-\frac{1}{64}} = \frac{4}{3}\pi \cdot \frac{64}{63}.$$

On the other hand, the volume of the cone is $\frac{1}{3}\pi(2)^2 \cdot \frac{8}{3} = \frac{4}{3}\pi \cdot \frac{8}{3}$. Thus, the volume of that portion of the cone lying outside all the spheres is

$$\frac{4\pi}{3}\left(\frac{8}{3} - \frac{64}{63}\right) = \frac{4\pi}{3} \cdot \frac{104}{63} = \frac{416\pi}{189}\,.$$

Problem 2. (Sudoku) A 4×4 Sudoku puzzle is a 4×4-square filled with numbers 1,2,3,4 so that the numbers in each row, column, and each of the four 2×2-square do not repeat. (You may already be familiar with a more complicated but similar 9×9 Sudoku puzzle.)

1	2	3	4
3	4	1	2
2	1	4	3
4	3	2	1

How many 4×4 puzzles are there? (Puzzles that are obtained from each other by rotation, reflection, etc. count as *different* puzzles.)

Answer. 288

Solution. We will count the puzzles up to symmetries, so that the number of possibilities becomes small, and then multiply this number by the number of symmetries to obtain the answer.

The first choice we can make is to choose the numbers in the first square to be 1,2,3,4 (we will then have to multiply the final answer by $4! = 24$).

1	2		
3	4		

Then we can choose the final two numbers in the first row to be 3,4 and the remaining numbers in the first column are 2,4 (we will multiply by 2×2 in the end).

1	2	3	4
3	4		
2			
4			

Then 1 and 4 are the only possible entries in the diagonal position $(3, 3)$ and it is easily seen that the choice of 1 leads to a contradiction. This leads to the following arrangements.

1	2	3	4
3	4		
2		4	
4			1

1	2	3	4
3	4		
2		4	
4			2

1	2	3	4
3	4		
2		4	
4			3

Each of the three then can be filled uniquely (do it!). So the final answer is

$$3 \cdot 4! \cdot 2 \cdot 2 = 288$$

Problem 3. (Coin flipping) A person is flipping a coin many times. With probability 1/3 each of the following can happen on each flip: heads comes up, tails comes up, or she gets bored and decides to stop. What is the probability that, after the game has stopped, there were never two heads in a row?

Answer. 4/5

Solution. Denote heads by H, tails by T, end by E.

First solution: Let p be the probability. Then we have the following equation:

$$p = \frac{1}{3} + \frac{1}{9} + \frac{1}{3}p + \frac{1}{9}p$$

explained as follows. If there were no heads in a row then the following could have happened:

1. End right away, with probability 1/3.
2. Heads and then End, with probability 1/9.
3. Tails, and then any sequence of plays without two Heads in a row, with probability $p/3$.
4. Heads, Tails, and then any sequence of plays without two Heads in a row, with probability $p/9$.

Solving this equation gives $p = 4/5$.

Second solution: Denote by A_n the number of sequences of n heads and tails without two heads in a row. Then what we have to find the sum

$$p = \frac{1}{3}A_0 + \frac{1}{3^2}A_1 + \frac{1}{3^3}A_2 + \ldots = \sum_{n=0}^{\infty} \frac{1}{3^{n+1}} A_n$$

What are A_n? $A_0 = 1$: no Heads or Tails, End right away. $A_1 = 2$: H, T. $A_2 = 3$: HT, TH, TT. $A_3 = 5$: HTH, HTT, THT, TTH, TTT.

Do you see any pattern? We have $A_n = A_{n-1} + A_{n-2}$. Indeed, any sequence can start with either HT and then any sequence of length $n-2$ without HH, or with T and then any sequence of length $n-1$ without HH. So A_n are the Fibonacci sequence shifted by 2: $A_n = F_{n+2}$, where F_0, F_1, etc. are 0, 1, 1, 2, 3, 5, 8, 13, 21, etc. (Note that this argument is equivalent to the observation that any positive integer can be written uniquely as the sum of non-consecutive Fibonacci numbers.)

The explicit formula for the n-th Fibonacci number is

$$F_n = \frac{1}{\sqrt{5}}(\phi^n - \hat{\phi}^n),$$

where $\phi = (1+\sqrt{5})/2$ is the golden ratio, and $\hat{\phi} = (1-\sqrt{5})/2$ is the negative of its inverse ($\phi\hat{\phi} = -1$).

Plugging in, and using the formula for the sum of the geometric series, we get

$$\frac{3}{\sqrt{5}} \sum_{n=0}^{\infty} \left(\left(\frac{\phi}{3}\right)^{n+2} - \left(\frac{\hat{\phi}}{3}\right)^{n+2} \right) = \frac{1}{\sqrt{5}} \left(\frac{\phi^2/3}{1-\phi/3} - \frac{\hat{\phi}^2/3}{1-\hat{\phi}/3} \right)$$

Simplifying and using $\phi + \hat{\phi} = 1$, $\phi - \hat{\phi} = \sqrt{5}$, $\phi\hat{\phi} = -1$, we get 4/5.

Authors. (1) was written by Ted Shifrin, (2) and (3) by Valery Alexeev and Boris Alexeev.

Sources. The source for (2) is A.M. Herzberg and M.R. Murthy, *Notices of the American Mathematical Society*, 54 (2007), 708–717.