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Team Round / 45 min / 210 points
October 13, 2007

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1. (Platonic solids and plane tilings) How many triples of positive integers (p, q, r) are there such that

(a) Each of p, q, and r is at least 2,

(b) At most one of p, q, and r equals 2, and

(c)
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \geq 1?$$

Note that in this problem, triples are *unordered*; for example, (p, q, r) and (q, p, r) count as different triples if $p \neq q$.

Answer. 25

Solution. Let us first find triples where $p \leq q \leq r$ and then count how many times they contribute because of different orderings. We perform some basic casework, beginning by running over p in increasing order. Note that the smallest possible value of p is 2 by condition (a). On the other hand, since $p \leq q \leq r$, we have

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{p} + \frac{1}{p} + \frac{1}{p} = \frac{3}{p},$$

so $p \leq 3$ or else we violate condition (c).

- Suppose $p = 2$. Then $q \geq 3$ by condition (b). On the other hand, since $q \leq r$, we have
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq \frac{1}{2} + \frac{2}{q},$$
so $q \leq 4$ or else we violate condition (c).

- Suppose $q = 3$. Then $3 \le r \le 6$, giving four solutions: $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$, and $(2, 3, 6)$.
- Suppose $q = 4$. Then $r = 4$, giving one solution: $(2, 4, 4)$.

- Suppose $p = 3$. Then, since $q \le r$, we have

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \le \frac{1}{3} + \frac{2}{q},$$

meaning that $q \le 3$ or else we violate condition (c). On the other hand, $q \ge 3$ since we assumed $p \le q$. But if $q = 3$, then $r = 3$ as well, so we have one solution: $(3, 3, 3)$.

Thus there are 6 ordered solutions. Now we count orderings.

The three triples $(2, 3, 4)$, $(2, 3, 5)$, and $(2, 3, 6)$ can each be ordered in six ways. The two triples $(2, 3, 3)$ and $(2, 4, 4)$ can each be ordered in three ways. Finally, the triple $(3, 3, 3)$ has only one ordering.

All in all, we have $3 \cdot 6 + 2 \cdot 3 + 1 = 18 + 6 + 1 = 25$ triples.

Connections. These triples appear in amazingly many places in mathematics. To note a few: in the classification of groups generated by reflections (Coxeter diagrams), in the classification of Lie algebras and Lie groups (Dynkin diagrams), in algebraic geometry (Du Val singularities), in representation theory (quivers), etc.

They also appear in the classification of regular polytopes. Indeed, the five solutions of the inequality

$$\frac{1}{2} + \frac{1}{q} + \frac{1}{r} > 1$$

correspond to the five platonic solids: the tetrahedron (3,3), the cube (4,3), the octahedron (3,4), the dodecahedron (5,3), and the icosahedron (3,5). Each pair (q, r) corresponds to the platonic solid whose faces are regular q-gons, with r of them meeting at every vertex.

The three solutions of the equality

$$\frac{1}{2} + \frac{1}{q} + \frac{1}{r} = 1$$

correspond to the tilings of the plane by regular polygons: by regular triangles (3,6), by squares (4,4), and by regular hexagons (6,3). Each pair (q, r) corresponds to the tiling by regular q-gons, with r of them meeting at every vertex.

(This explains the mysterious title of the problem.)

Problem 2. (M-triples) How many triples of *positive integers* (a, b, c) are there with $a \le b \le c \le 100$ that satisfy the equation $a^2 + b^2 + c^2 = 3abc$?

Note that in this problem, triples are *ordered*; we insist that $a \le b \le c$.

Answer. 7

Solution. One first notes that $(1,1,1)$ is a solution. One also notes that fixing a and b, the above equation is a quadratic equation in the unknown variable c. By Viète's theorem, the sum of roots of the quadratic equation

$$c^2 - 3ab \cdot c + (a^2 + b^2) = 0$$

is $3ab$. If c is a solution then so is $3ab - c$. In other words, if the triple (a, b, c) is a solution then so is the triple $(a, b, 3ab - c)$. One can also fix a, c and change b; or fix b, c and change a.

After changing, we reorder so that we again have $a \leq b \leq c$. Let us call this rule for producing new solutions from known solutions a *mutation.*

Beginning with the solution $(1,1,1)$, one can successfully mutate to

$$\begin{tikzcd} (1,1,1) \arrow[r, leftrightarrow, "1", "2"'] & (1,1,2) \arrow[r, leftrightarrow, "1", "5"'] & (1,2,5) \arrow[r, leftrightarrow, "2", "13"'] \arrow[d, leftrightarrow, "1"', "29"] & (1,5,13) \arrow[r, leftrightarrow, "5", "34"'] & (1,13,34) \arrow[r, leftrightarrow, "13", "89"'] & (1,34,89) \\ & & (2,5,29) & & & \end{tikzcd}$$

Mutating further produces numbers greater than 100.

Now we prove that that these are actually all the solutions. For this, we show that for any solution triple $(a, b, c) \neq (1,1,1)$, with $a \leq b \leq c$, one has $3ab - c < c$. This implies that any solution triple with $a, b, c \leq 100$ can be mutated to a *smaller* triple. Hence, repeating this process we eventually get to $(1,1,1)$. Therefore, *any* solution can be obtained by mutations starting from $(1,1,1)$ and going up.

So, we want to show that if (a, b, c) is a solution different from $(1,1,1)$ then $3ab - c < c$, i.e. $3ab < 2c$, or $3abc < 2c^2$. Since $3abc = a^2 + b^2 + c^2$, this is equivalent to showing that $a^2 + b^2 < c^2$.

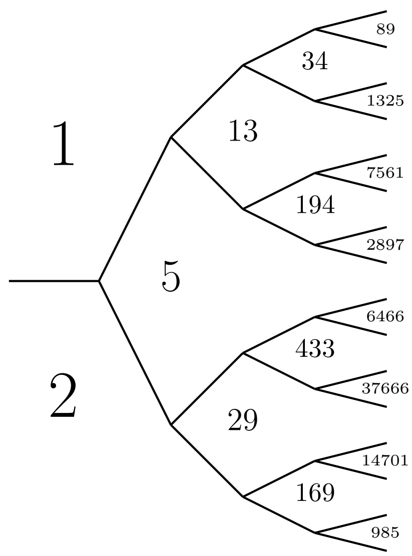
If $a \leq b \leq c$ then $a^2 + b^2 + c^2 \leq 3c^2$. So we have $3abc \leq 3c^2$. Thus, $ab \leq c$ and $a^2b^2 \leq c^2$.

The inequality $a^2b^2 > a^2 + b^2$ is equivalent to $(a^2 - 1)(b^2 - 1) > 1$. This always holds unless $a = 1$ or $b = 1$ (but then also $a = 1$ since $a \leq b$). So if $a \neq 1$ we are already done: we have $a^2 + b^2 < a^2b^2 \leq c^2$ as desired.

Now suppose $a = 1$. We must show that $1 + b^2 < c^2$. The only way this may not be true is if $b = c$. But then we get the equation $1 + 2b^2 = 3b^3$, which has no solution with $b > 1$. Indeed, $b > 1$ implies $b^3 > 1$ and $b^3 > b^2$. So we are done.

History: Such triples are commonly known as *Markov triples.* They were discovered by a Russian mathematician Andrey Markov in the 19th century. They appear in many fields of mathematics, from statistics to algebraic geometry.

Another way to visualize Markov triples is through a binary tree as in the diagram below, wherein every vertex corresponds to the Markov triple consisting of the numbers on the three adjacent regions. The triples $(1,1,1)$ and $(1,1,2)$ are not directly represented.



Notice that the numbers adjacent to 1 are Fibonacci numbers, skipping every other one, while the numbers adjacent to 2 are Pell numbers (also skipping every other one).

Problem 3. (Holy polyhedron) There exists a (non-convex) polyhedron with exactly one hole, such that every pair of faces shares exactly one edge. How many vertices does this polyhedron have?

Answer. 14

Solution. Suppose this polyhedron has F faces, E edges, and V vertices. Recall that the generalization of Euler's formula $F + V - E = 2$ for polyhedra with holes is

$$F + V - E = 2 \cdot (1 - H),$$

where H is the number of holes. In our case, $H = 1$ so

$$F + V = E.$$

(See the note at the end of the proof)

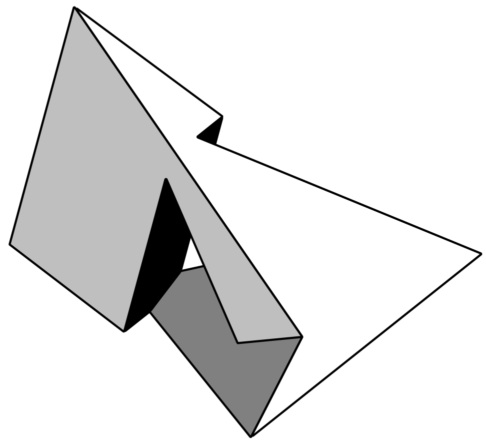
Furthermore, each pair of faces meets in exactly one edge, so

$$E = \binom{F}{2}.$$

Finally, at every vertex, exactly three faces meet. Indeed, otherwise two of the faces that meet there couldn't be adjacent! By counting vertex-edge incidences (three edges meet at each vertex, and two vertices form the endpoints of each edge), we have

$$3V = 2E.$$

By solving this system of equations (for example, by plugging the first and last equation into the second to get $3F = \binom{F}{2}$), we find that $F = 7$, $E = 21$, and $V = 14$.



This polyhedron was discovered by Lajos Szilassi in 1977, and is dual to the Császár polyhedron discovered in 1949, which has one hole and “no diagonals”. It is unknown whether or not any more polyhedra exist of either type (all pairs of faces joined by an edge, or all pairs of vertices connected by an edge) other than these two and the tetrahedron. By following an analysis like that above, we can see that the next Szilassi-type polyhedron must have 6 holes, 12 faces, 66 edges, and 44 vertices.

Note The usual proof of Euler’s formula is by induction on the *graph* of the polyhedron. The same works here, except the base case is different. It is not difficult to check that for any graph on a torus, the above relation holds. Recall, however, that *face* in the above discussion means a region similar to one on the plane; in particular, the empty graph on the torus does not have any well-defined faces. However, arguably the smallest graph that does has a single vertex and two edges. The two edges wrap all the way around the torus in different directions, cutting the torus into a single well-defined face. In this case, $F + V = 1 + 1 = 2 = E$, so the same is true of all graphs on the torus. A similar graph exists with one vertex, $2H$ edges, and one face for any number of holes H.

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Sources. All of the problems are original. Problems 1 and 2 are based on well-known mathematics, while Problem 3 was inspired by an email from Steve Sigur.