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TEAM ROUND / 45 MIN / 210 POINTS
October 17, 2009

**WITH SOLUTIONS**

**No calculators are allowed on this test.** You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

**Problem 1. (Up or down?)** What is the smallest $n$ such that in any sequence of $n$ distinct numbers

$$a_1, a_2, a_3, \dots, a_n$$

there is either an increasing subsequence of length 10 or a decreasing subsequence of length 10?

*Example.* In the sequence

$$1, 3, 2, 8, 4, 0, 5$$

a longest increasing sequence has length 4:

$$\boxed{1}, 3, \boxed{2}, 8, \boxed{4}, 0, \boxed{5}$$

and a longest decreasing sequence has length 3:

$$1, \boxed{3}, \boxed{2}, 8, 4, \boxed{0}, 5$$

**Answer.** 82

**Solution.** *Hint:* you can guess the answer by looking at the easier cases of sequences of lengths 2, 3, 4. . . . The answers are 2, 5, 10. . .

Here is a sequence of longest possible length 81 which *does not* have increasing or decreasing subsequences of length 10:

$$9, 8, \ldots, 1,\ 18, 17, \ldots, 10,\ 27, 26, \ldots, 19, \ldots, 81, 80, \ldots, 73.$$

But we claim that every sequence of length $\geq 82$ will have a required subsequence. Here is why. To every element $a_i$ of the sequence, assign a pair of numbers $(b_i, c_i)$ of positive integers as follows:

1. $b_i$, the length of the longest increasing sequence ending with $a_i$, and
2. $c_i$, the length of the longest decreasing sequence ending with $a_i$.

We claim that for different $a_i$ and $a_j$, the pairs are different.

Indeed, let us say $i < j$. If $a_i < a_j$ then the longest increasing sequence for $a_j$ is guaranteed to be longer than the one for $a_i$, so $b_j > b_i$. Similarly, if $a_i > a_j$ then $c_j > c_i$.

If all increasing sequences have lengths $\leq 9$ then for the pairs $(b_i, c_i)$ there are only $9^2 = 81$ possibilities. So with 82 numbers, we are guaranteed to have a sequence of length 10.

**Problem 2. (Doubling up)** Find a positive integer which doubles when its last digit is moved in front. The number is to be written in standard decimal notation with no leading zeroes.

*Example.* The number 1234 becomes 4123 when its last digit is moved in front (so it doesn't work).

**Answer.**

$$105263157894736842$$

is the smallest. One can obtain every such number from this by rotating the digits (in such a way that the next to the last digit is even) or by repeating the entire number multiple times, *e.g.*

$$947368421052631578947368421052631578.$$

(So the number of digits is divisible by 18.)

**Solution.** *First solution.* Let $x$ be such a number, with last digit $d$ and $n$ digits total. Then by construction, $\frac{x-d+10^n d}{10} = 2x$, or in other words

$$19x = (10^n - 1)d.$$

Thus we seek an $n$ such that $10^n - 1$ is divisible by 19. By trying successive $n$ or by Fermat's little theorem, $n = 18$ works. Setting $d = 1$ results in a leading 0, but $d = 2$ gives

$$\frac{10^{18}-1}{19} \cdot 2 = 105263157894736842.$$

*Second solution.* One could also construct this number as follows.

You want the last digit as small as possible, so that the doubled number will start with a smal digit. But the last digit can't be 0 – moving that to the front makes the number smaller – and it can't be a 1 – a doubled number cannot begin with a 1 unless it has more digits than the number that was doubled. So begin with last digit 2. Moving this to the front is supposed to double the number $V$, so the last digit of the doubled number $2N$, and thus next to the last digit of $N$, is 4. Repeating this, we get

$$N = \ldots 6842$$

At this point we have to start paying attntion to carries: $2 \times 842 = 1684$, so the next digit must be 3 ($= 2 \times 6 + 1 \mod 10$). Continuing, we get

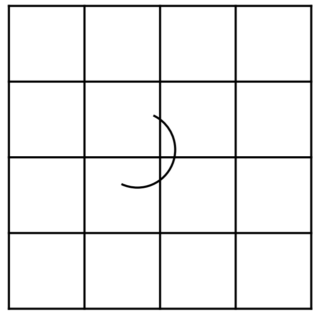
$$N = 105263157894736842$$

We could continue, but we would only repeat the same digits.

Beginning with any other digit $d > 2$ will generate the same set of digits as above, beginning with $d$ which is preceded by an even digit.

**Problem 3. (Lucky horseshoe)** A horseshoe has the shape of a semicircle of diameter 1. We throw it randomly on a square grid of 1 by 1 squares and count how many times it intersects the lines. After a very large number $N$ of throws, the number of intersections will be close to $cN$ for some number $c$. What is $c$?

*Example.* The following throw counts for three intersections:



**Answer.** 2

**Solution.** If we throw a *circle* of diameter 1, then after $N$ times, we expect approximately $4N$ intersections. Indeed, the circle will have exactly 4 intersections each time, except in the impossibly unlikely event that it hits one of the grid's intersection points. Two semicircles make a circle, so for each of them the number of intersections will be $2N$. So $c = 2$.

In general, for an arbitrary curve of length $\ell$, one has $c = 4\ell/\pi$. To prove this, note that for the mathematical expectation (that is what we are computing here) one has

$$E(\xi_1 + \cdots + \xi_n) = E\xi_1 + \cdots + E\xi_n$$

Therefore, we can slice our curve into small pieces and make other curves from these pieces. If our curve has a length then it can be approximated by a sequence of straight segments, by which we can then approximate another curve. Then the above formula shows that the answer is proportional to the length of the curve. For a circe of diameter 1 and length $\pi$, the answer is 4. So the coefficient of proportionality is $4/\pi$.

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**Sources.** (1) is a special case of a theorem of Erdős-Szekeres, see `http://en.wikipedia.org/wiki/Erdos-Szekeres_theorem`. For (2), see this New York Times article about Freeman Dyson: `http://www.nytimes.com/2009/03/29/magazine/29Dyson-t.html`, last page. Gil Kalai recently blogged about problem (3) on his blog "Combinatorics and more": see the entry on `http://gilkalai.wordpress.com/2009/08/03/` about Buffon's needle and Buffon's "noodle".