1785

Sponsored by: UGA Math Department and UGA Math Club

Team Round / 1 hour / 210 points
October 20, 2012

WITH SOLUTIONS

No calculators are allowed on this test. You do not have to provide proofs; only the answers matter. Each problem is worth 70 points, for a total of 210 points.

Problem 1 (Crazy dice)**.** An eccentric friend of yours has a pair of fair, six-sided dice, all of whose faces are labeled with positive integers. When the two dice are rolled, the sum of the top faces has the same probability distribution as that of a standard pair of six-sided dice. In other words, the possible outcomes range from 2 to 12, and each of these occurs with the same probability as for a standard pair of dice.

However, your friend's dice are **not** a standard pair; in fact, the faces of the first die are labeled $1, 2, 2, 3, 3, 4$. What are the labels on the faces of the other die? **Write them down in increasing order.**

Answer. $1, 3, 4, 5, 6, 8$.

Solution. In fact, our solution will show how to find the labels on the dice *without* being given either set of labels in advance.

Suppose that the faces of the two dice are labeled $a_1, \dots, a_6$ and $b_1, \dots, b_6$. Since the probability distribution of the sum of top faces agrees with that of the standard dice, we find that we have a polynomial identity

$$(z^{a_1} + z^{a_2} + \cdots + z^{a_6})(z^{b_1} + z^{b_2} + \cdots + z^{b_6}) = (z^6 + z^5 + \cdots + z)^2. \qquad (1)$$

(If you don't already see what is going on, multiply out the right-hand side and compare your answer with the entries in the probability distribution table.) Factoring as far as possible, we find that

$$\begin{aligned} z + z^2 + \cdots + z^6 &= z(z^5 + z^4 + \cdots + z + 1) \\ &= z(z+1)(z^4 + z^2 + 1) \\ &= z(z+1)((z^2+1)^2 - z^2) \\ &= z(z+1)(z^2 - z + 1)(z^2 + z + 1). \end{aligned}$$

So for (1) to hold, it must be that

$$z^{a_1} + z^{a_2} + \cdots + z^{a_6} = z^{\epsilon_1}(z+1)^{\epsilon_2}(z^2 - z + 1)^{\epsilon_3}(z^2 + z + 1)^{\epsilon_4}$$

while

$$z^{b_1} + z^{b_2} + \cdots + z^{b_6} = z^{\epsilon_1'}(z+1)^{\epsilon_2'}(z^2 - z + 1)^{\epsilon_3'}(z^2 + z + 1)^{\epsilon_4'},$$

for nonnegative integer exponents ϵ_i and ϵ_i' satisfying $\epsilon_i + \epsilon_i' = 2$. Since all the numbers on both dice are *positive* integers, it must be that $\epsilon_1 = \epsilon_2 = 1$. Plugging in $z = 1$, the first of the above two equations gives us $6 = 2^{\epsilon_2} \cdot 3^{\epsilon_4}$, and the second gives us $6 = 2^{\epsilon_2'} 3^{\epsilon_4'}$. So $\epsilon_2 = \epsilon_2' = 1$ and $\epsilon_4 = \epsilon_4' = 1$. It remains to figure out ϵ_3 and ϵ_3'. If $\epsilon_3 = \epsilon_3' = 1$, then we recover the standard dice. The only remaining possibility is that one is 0 and the other is 2; say $\epsilon_3 = 0$ and $\epsilon_3' = 2$. We then find that

$$\begin{aligned} z^{a_1} + z^{a_2} + \cdots + z^{a_6} &= z(z+1)(z^2 + z + 1) \\ &= z^4 + z^3 + z^3 + z^2 + z^2 + z \end{aligned}$$

and

$$\begin{aligned} z^{b_1} + z^{b_2} + \cdots + z^{b_6} &= z(z+1)(z^2 - z + 1)^2(z^2 + z + 1) \\ &= z^8 + z^6 + z^5 + z^4 + z^3 + z; \end{aligned}$$

this shows that the "crazy dice" are labeled $4, 3, 3, 2, 2, 1$ and $8, 6, 5, 4, 3, 1$.

These crazy dice were discovered by Colonel George Sicherman of Buffalo, New York. They were first reported on by Martin Gardner, in a February, 1978 column in *Scientific American*. The argument with polynomials given above is due to the mathematicians Joseph Gallian and Duane Broline.

Here is a further problem which you might enjoy investigating: Is there a pair of 4-sided crazy dice? 5-sided? 7-sided? In general, for which n does an n-sided pair exist? (This problem is solved in a paper of David Rusin and Joseph Gallian: Cyclotomic polynomials and nonstandard dice, Discrete Math. 27 (1979), no. 3, pp. 245–259.)

Problem 2 (Squares)**.** It is known that there exists a unique positive integer $n > 1$ such that

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = m^2$$

is a square of another integer m. **Find n.**

Answer. 24

Solution. The easiest (although laborious) solution is just to keep adding squares until the sum is a square number:

$$1^2 + 2^2 + 3^2 + \ldots + 24^2 = 70^2$$

A smarter solution is to use the formula

$$1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6},$$

which means that we need to solve the equation $n(n+1)(2n+1) = 6m^2$. Noting that n, $n+1$ and $2n+1$ are pairwise relatively prime, this means that the numbers

$$n,\ n+1,\ 2n+1$$

should be of one of the following shapes, for some integers a, b, c:

$$6a^2, b^2, c^2 \qquad a^2, 6b^2, c^2 \qquad a^2, b^2, 6c^2 \qquad 2a^2, 3b^2, c^2, \qquad 2a^2, b^2, 3c^2, \qquad \text{etc.}$$

A quick search finds $n = 6 \cdot 2^2$, $n + 1 = 5^2$, $2n + 1 = 7^2$.

This problem is known as Lucas square cannonball problem, since it can be visualized as the problem of taking a square arrangement of cannonballs on the ground and building a square pyramid out of them. Edouard Lucas is the same guy who invented the famous Tower of Hanoi puzzle that you undoubtedly played as children.

If you are interested, a complete solution to the above problem can be found in `http://thales.doa.fmph.uniba.sk/sleziak/vyuka/2010/semtc2/clanky/AnglinSquarePyramid.pdf`.

This problem is remarkable also in the fact that it appears in the construction of the famous Leech lattice, an integral even unimodular lattice of dimension 24 with shortest nonzero integral vector of length 2.

Indeed, the Leech lattice can be constructed as $e^{\perp}/e$, where $e = (0, 1, \ldots, 24; 70)$ is a self-orthogonal vector in the Lorentzian lattice $\mathbb{Z}^{25,1}$ in 26-dimensional Lorentzian space-time $\mathbb{R}^{25,1}$. This model of the universe appears in some variants of string theory in physics. It has 25 spatial dimensions instead of the three we commonly perceive.

Problem 3 (Exciting tournaments)**.** Sixteen ($16 = 2^4$) teams participate in a single-elimination tournament of four rounds. That is, each team plays some other team in the first round, and the winners advance to the second round, and so forth. Suppose the teams are currently ranked best to worst from #1 to #16 and the higher-ranked team always wins in every game. The tournament designers, instead of using the rankings, chose the team matchups completely randomly. What is the probability that teams #1–#8 all advance to the quarterfinals (second round), teams #1–#4 all advance to the semifinals (third round), and teams #1 and #2 meet in the final (fourth round)?

Express your answer as a fraction in simplified form. (The graders will not do arithmetics for you.)

Answer.

$$\frac{2048}{675675}$$

Solution. Call a system of team matchups *exciting* if it has the desired property of highly-ranked teams meeting late in the tournament.

One solution is by directly counting all possible matchups versus exciting matchups. Suppose we have a "bracket" of sixteen teams drawn out, where we *distinguish* all sixteen possible starting positions. (For example, if you keep all of the matchups the same, but switch the "positions" of two teams playing each other, this counts as a *different* matchup.)

Then there are clearly $16!$ total matchups. To count the exciting matchups, first consider the top-ranked team. It can be placed in any of 16 possible locations. The #2 team, however, must be placed in the opposite half of the bracket, and so may be placed in any of 8 possible locations. The #3 and #4 teams must be allocated one each to different halves of the brackets, in two possible ways; once the halves are chosen, they must appear in the opposite quarter of the bracket from the higher ranked #1 and #2 teams. There are thus 4 possible locations for each of them. Similar reasoning for the #5–#8 teams gives 4! different pairings with higher-ranked teams, and 2^4 possible locations. Finally, we get 8! possibilities for the last eight teams. All in all, there are

$$(16) \cdot (8) \cdot (2! \cdot 4^2) \cdot (4! \cdot 2^4) \cdot (8!)$$

exciting brackets.

The desired probability is

$$\frac{(16) \cdot (8) \cdot (2! \cdot 4^2) \cdot (4! \cdot 2^4) \cdot (8!)}{16!} = \frac{2048}{675675},$$

where we find that many terms cancel. Knowing the curious fact that $7 \cdot 11 \cdot 13 = 1001$ helps with the arithmetic.

Alternative solution: If we consider the only important feature of a tournament which teams play each other, and not their *positions* in a bracket, then we would find there to be

$$\frac{16!}{2^{15}}$$

total tournaments and

$$2! \cdot 4! \cdot 8!$$

exciting tournaments.

Third solution: The above solutions can be directly reinterpreted probabilistically. Specifically, the #2 team meets the #1 team in the final with probability $\frac{8}{15}$. Assuming the final is correct, the #3 team makes it to the semifinal with probability $\frac{8}{14}$. Assuming that works out, the #4 team is fine with probability $\frac{4}{13}$. Writing out all of the probabilities gives

$$\frac{16}{16} \cdot \frac{8}{15} \cdot \frac{8}{14} \cdot \frac{4}{13} \cdot \frac{8}{12} \cdot \frac{6}{11} \cdot \frac{4}{10} \cdot \frac{2}{9} = \frac{2048}{675675}.$$

There are also some other ways of looking at the problem, particularly starting from the #16 team instead of the #1 team.

Authors. Problem 1 was written by Paul Pollack, problem 2 by Valery Alexeev, and problem 3 by Boris Alexeev.