**Problem 1** (Lights out). In the game of “Lights out” there is a collection of lights, some of which are on, and some off. If you touch any light, that light and all of the adjacent lights will change; those that are on will turn off, and those that are off will turn on. You win if you can touch a sequence of lights so that all of the lights are off. Depending on which lights are initially on, this may not be possible.

For this problem, there are 8 lights, located at the corners of a cube, so that each light is adjacent to 3 other lights. How many “winning positions” are there? In other words, for how many initial configurations of on/off lights is it possible to turn off all of the lights? Having all of the lights initially off counts as a winning position (you’ve already won!).

**Answer.** 16

**Solution.** For reference, we label the vertices of the cube as shown.

First, some observations:
1. The status of the lights can be represented by the 8-dimensional column vector $\mathbf{a} = (a_1, a_2, \ldots, a_8)^T$ with $a_i = 1$ if the $i$th light is on and $a_i = 0$ if the $i$th light is off.

2. We can think of the components of this vector as belonging to the field $\mathbb{F}_2$ containing only the two elements 0 and 1; in this field, $1 + 1 = 0$.

3. Touching a light has the effect of adding a vector $\mathbf{v}_i$ to the current status vector. The vector to be added has 1s in the 4 positions corresponding to the vertex touched and its 3 adjacent vertices; the other 4 positions are 0; e.g., $\mathbf{v}_1 = (1, 1, 0, 1, 0, 0, 0, 0)^T$.

4. Since vector addition is commutative ($\mathbf{v}_i + \mathbf{v}_j = \mathbf{v}_j + \mathbf{v}_i$), it follows from observation (3) that the order in which the lights are touched is irrelevant. Moreover, since $\mathbf{v}_i + \mathbf{v}_i = \mathbf{0}$, no light need ever be touched twice.

5. From the last observation in (4), if $(a_1, a_2, \ldots, a_8)^T$ is a “winning position,” then you can reach $(0, 0, \ldots, 0)^T$ by touching at most 8 lights; we can represent the lights touched by a column vector $(b_1, \ldots, b_8)^T$, where $b_i = 1$ if the light is touched and $b_i = 0$ otherwise.

6. If you start in position $(0, 0, \ldots, 0)^T$, then that same sequence of light touches represented by $(b_1, \ldots, b_8)^T$ will return you to $(a_1, \ldots, a_8)^T$; i.e., the winning positions are those that can be reached in eight or fewer light touches.

7. Thus, the winning positions are the linear combinations of the vectors $\mathbf{v}_i$ with coefficients from $\mathbb{F}_2$. In other words, they are the vectors that belong to the column space of

$$V = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \end{bmatrix}.$$  

(Look carefully and you’ll see this matrix hidden on this year’s T-shirt!)
8. The dimension of the column space (over $\mathbb{F}_2$) is the rank of the matrix, which we claim is 4 (see below). So the number of winning positions is $2^4 = 16$.

To see that the rank is 4 you can simply row reduce the matrix. The $4 \times 4$ identity matrix in the lower left corner shows that the rank is at least 4; on the other hand, the row operations

$$
R_1 \mapsto R_1 + (R_5 + R_6 + R_8) = 0 \\
R_2 \mapsto R_2 + (R_5 + R_6 + R_7) = 0 \\
R_3 \mapsto R_3 + (R_6 + R_7 + R_8) = 0 \\
R_4 \mapsto R_4 + (R_5 + R_7 + R_8) = 0
$$

show that rank $\leq 4$.

So what are the winning positions? First notice that

$$
\begin{align*}
\mathbf{v}_5 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_4 \\
\mathbf{v}_6 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \\
\mathbf{v}_7 &= \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \\
\mathbf{v}_8 &= \mathbf{v}_1 + \mathbf{v}_3 + \mathbf{v}_4
\end{align*}
$$

so any position can be reached using only lights 1, 2, 3, and 4. The $2^4$ combinations of these give the different winning positions.

**Problem 2** (Counting lattice points). Partition the plane into $1 \times 1$ squares using the lines $x = n$ and $y = m$ for all integers $m$ and $n$. Then draw a circle of radius 100 centered at (0, 0). How many of the $1 \times 1$ squares does the circle pass through the interior of? Notice that the circle passes through the interior of the square whose lower left corner is (0, 99), but it does not pass through the interior of the square whose lower left corner is (0, 100).

**Answer.** 780

**Solution.** We begin by counting the squares in the first quadrant between the lines $y = x$ and the positive $y$-axis. In this half of the quadrant, the tangent lines to the circle have slopes in $[-1, 0]$, and so the circle can pass through a square in these three ways:
but not this

Also note that this

is not passing through the interior of the square.

This shows that this piece of the circle intersects every column of boxes once, except for the columns in which the circle crosses a horizontal line. In that case, the circle passes through 2 boxes unless the circle crosses the line at a corner, in which case there is again 1 box in that column.

Notice that the line \( y = x \) intersects the circle \( x^2 + y^2 = 100^2 \) when \( x = \sqrt{5000} \), so \( 70 < x < 71 \); i.e., the circle passes through the interior of the square whose lower left corner is \( (70, 70) \).

To get from \( (0, 100) \) to \( (70, 70) \), the circle passes through 70 columns (those whose left sides range from \( x = 0 \) to \( x = 69 \)) which accounts for 70 squares. The circle also crosses 29 horizontal lines \( (y = 99 \rightarrow y = 71) \). Allowing for two Pythagorean triples \( (60, 80, 100) \) and \( (28, 96, 100) \) reduces this to 27 new squares. So we have a total of 97 squares in this half-quadrant; adding the 8 half-quadrants and the 4 squares lying on \( y = \pm x \), we get \( 8 \times 97 + 4 = 780 \) squares.

**Problem 3** (Primitive vertices). Let \( P_n \) denote the regular \( n \)-gon centered at the origin and having one vertex at \( (1, 0) \). We adopt the convention that \( P_1 \) consists of the single point \( (1, 0) \) and that \( P_2 \) consists of the line segment connecting \( (-1, 0) \) and \( (1, 0) \). A vertex of \( P_n \) is called *primitive* if it is not a vertex of \( P_m \) for any \( m < n \). For example, each vertex of \( P_3 \) is primitive except \( (1, 0) \). Let \( C_n \) denote the center of mass of the primitive vertices of \( P_n \). For how many \( n \leq 100 \) is \( C_n \) located at \( (0, 0) \)?

**Answer.** 39
Solution. By de Moivre’s theorem, the vertices \((x, y)\) of \(P_n\) correspond to the complex numbers \(\zeta = x + yi\) satisfying \(\zeta^n = 1\). A vertex of \(P_n\) is primitive if the corresponding value of \(\zeta\) does not satisfy \(\zeta^m = 1\) for any \(m < n\). Such a \(\zeta\) is called a primitive \(n\)th root of unity.

For each natural number \(n\), let \(U_n = \{\zeta \in \mathbb{C} : \zeta^n = 1\}\), and let \(V_n = \{\zeta \in \mathbb{C} : \zeta\) is a primitive \(n\)th root of unity\}. We are asked for the number of \(n \leq 100\) for which \(S(n) = 0\), where the function \(S\) is defined by

\[
S(n) = \sum_{\zeta \in V_n} \zeta.
\]

If \(\zeta \in U_n\), then \(\zeta \in V_m\) for a unique positive integer \(m\) dividing \(n\). Conversely, if \(m\) divides \(n\) and \(\zeta \in V_m\), then \(\zeta \in U_n\). Thus, \(U_n\) is the disjoint union of the sets \(V_m\) for \(m\) dividing \(n\). Note that as complex polynomials,

\[
x^n - 1 = \prod_{\zeta \in U_n} (x - \zeta).
\]

If \(n > 1\), comparing the coefficients of \(x^{n-1}\) yields \(\sum_{\zeta \in U_n} \zeta = 0\). Thus,

\[
S(1) = 1, \quad \text{and for all } n > 1, \quad \sum_{m \mid n} S(m) = 0. \quad (*)
\]

Here the sum is over all positive integers \(m\) dividing \(n\).

The relation (*) inductively determines all of the values of \(S(m)\). Moreover, we can compute enough examples to guess a formula for \(S(m)\). To start off, notice that if \(p\) is prime, then (*) shows \(S(1) + S(p) = 1 + S(p) = 0\), so \(S(p) = -1\). On the other hand, if \(n = pq\) is a product of two different primes, (*) gives \(0 = S(1) + S(p) + S(q) + S(pq) = 1 - 1 - 1 + S(pq)\), so that \(S(pq) = 1\). If \(n = p^2\) is the square of a prime, then \(0 = S(1) + S(p) + S(p^2) = 1 - 1 + S(p^2)\), and so \(S(p^2) = 0\). Continuing to experiment in this way, we arrive at the following guess:

\[
S(n) = (-1)^k \text{ if } n \text{ is a product of } k \text{ distinct primes, and } S(n) = 0 \text{ otherwise, that is, whenever } n \text{ has a repeated prime factor.}
\]

This guess is correct and we prove it below. Taking its correctness as given, we can complete our analysis of how often \(S(n) = 0\). This occurs exactly when \(n\) is divisible by \(p^2\) for some prime \(p\). Since \(n \leq 100\), the only possibilities for \(p\) are 2, 3, 5, or 7. Moreover, the only products of two primes
whose squares can divide a number \( n \leq 100 \) are 6 and 10, and no \( n \leq 100 \) can be divisible by the square of three different primes. So by inclusion-exclusion, the desired count of \( n \) is

\[
\left\lfloor \frac{100}{2^2} \right\rfloor + \left\lfloor \frac{100}{3^2} \right\rfloor + \left\lfloor \frac{100}{5^2} \right\rfloor + \left\lfloor \frac{100}{7^2} \right\rfloor - \left\lfloor \frac{100}{6^2} \right\rfloor - \left\lfloor \frac{100}{10^2} \right\rfloor = 25 + 11 + 4 + 2 - 2 - 1 = 39.
\]

**Proof of our guess.** We use the method of generating functions. Given an infinite sequence of complex numbers \( \{a_n\}_{n=1}^{\infty} \), we define the associated formal Dirichlet series by the expression \( \sum_{n=1}^{\infty} a_n n^{-s} \). We multiply two formal Dirichlet series by the rule

\[
\left( \sum_{n=1}^{\infty} a_n n^{-s} \right) \left( \sum_{n=1}^{\infty} b_n n^{-s} \right) = \sum_{n=1}^{\infty} \left( \sum_{d \geq 1, e \geq 1 \text{ s.t. } de = n} a_d \cdot b_e \right) n^{-s}
\]

\[
= \sum_{n=1}^{\infty} \left( \sum_{d | n} a_{n/d} \cdot b_d \right) n^{-s}.
\]

In the language of formal Dirichlet series, (*) says that

\[
\left( \sum_{n=1}^{\infty} n^{-s} \right) \left( \sum_{n=1}^{\infty} S(n) n^{-s} \right) = 1,
\]

and thus

\[
\sum_{n=1}^{\infty} S(n) n^{-s} = \frac{1}{\sum_{n=1}^{\infty} n^{-s}}.
\]

Because each positive integer can be factored uniquely as a product of primes,

\[
\sum_{n=1}^{\infty} n^{-s} = (1 + 2^{-s} + 2^{-2s} + \ldots)(1 + 3^{-s} + 3^{-2s} + \ldots)(1 + 5^{-s} + 5^{-2s} + \ldots)
\]

\[
= \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.
\]

(This famous identity due to Leonhard Euler plays an important role in
Taking reciprocals, we find

\[
\sum_{n=1}^{\infty} S(n) n^{-s} = \frac{1}{\sum_{n=1}^{\infty} n^{-s}} = \prod_{p \text{ prime}} (1 - p^{-s}) = \sum_{n=1}^{\infty} \mu(n) n^{-s},
\]

where \( \mu(n) = (-1)^k \) if \( n \) is a product of \( k \) distinct primes, and \( \mu(n) = 0 \) otherwise. Comparing the coefficients of \( n^{-s} \) on both sides, we see \( S(n) = \mu(n) \) for all values of \( n \). This confirms the guess we made above.

Authors. Problems and solutions were written by Mo Hendon and Paul Pollack.

Sources. Problem #2 is adapted from a problem in Math Girls Talk about Trigonometry, to be released by Bento Books in late 2014.