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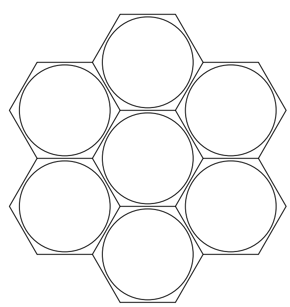
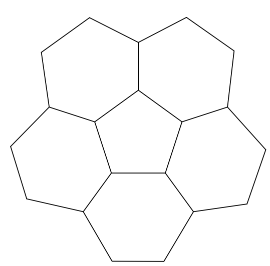
Team Round / 1 hour / 210 points
October 22, 2016

WITH SOLUTIONS

Problem 1 (How many pentagons?)**.** A golf ball is basically a sphere with a lot of circular dimples. Most of these are arranged in a hexagonal array like this:



But some dimples are pentagonal, surrounded by hexagons like this:



Assume that the golf ball is covered with these hexagonal and pentagonal dimples, and, as in the diagrams, three of these polygons meet at each vertex.

If there are 332 hexagons, how many pentagons are there?

Answer. 12 (pentagons)

Solution. Let H and P be the number of hexagons and pentagons, respectively. Then the number of vertices on the ball is

$$V = \frac{6H + 5P}{3},$$

the number of edges is

$$E = \frac{6H + 5P}{2},$$

and the number of faces is

$$F = H + P.$$

Euler's formula says that

$$V - E + F = 2,$$

and so

$$\frac{6H + 5P}{3} - \frac{6H + 5P}{2} + H + P = 2.$$

Multiply by 6:

$$(12H + 10P) - (18H + 15P) + 6H + 6P = 12.$$

Thus,

$$P = 12.$$

Note the surprise twist: The H's cancelled, and so the number of hexagons is irrelevant. That's why a soccer ball also has 12 pentagons!

Remark: The number 332 is taken from this *New Scientist* article:
`https://www.newscientist.com/article/dn1746-hexagonal-dimples-boost-golf-balls-flight/`

Problem 2 (Pascal sums)**.** The 2016th row of Pascal's triangle begins

$$1, \quad 2016, \quad 2031120, \quad 1363558560, \quad \ldots.$$

Let

$$S_1 = \text{the sum of every 4th number, beginning with 1,}$$
$$S_2 = \text{the sum of every 4th number, beginning with 2016,}$$
$$S_3 = \text{the sum of every 4th number, beginning with 2031120,}$$
$$S_4 = \text{the sum of every 4th number, beginning with 1363558560.}$$

What is the difference between the largest and smallest S_i? Write your answer in terms of powers of 2.

Solution. Questions about sums of entries in Pascal's triangle are often related to the binomial theorem. Let's begin by recalling some such summations:

- The sum of the entries in the 2016th row of Pascal's triangle is 2^{2016}; i.e., $\sum_{k=0}^{2016} \binom{2016}{k} = 2^{2016}$. You can see this by putting $x = y = 1$ in the binomial theorem:

$$(x+y)^{2016} = \sum_{k=0}^{2016} \binom{2016}{k} x^{n-k} y^k.$$

  In the language of this question, $S_1 + S_2 + S_3 + S_4 = 2^{2016}$.

- Similarly, the alternating sum of the entries in the 2016th row is 0: $\sum_{k=0}^{2016} (-1)^k \binom{2016}{k} = 0$. To see this, put $x = 1$ and $y = -1$ into the binomial theorem. Another way of saying this: The sum of the even-numbered entries equals the sum of the odd-numbered entries: $\binom{n}{0} + \binom{n}{2} + \cdots = \binom{n}{1} + \binom{n}{3} + \ldots$. This common sum must be 2^{2015}: their difference is zero, but their sum is 2^{2016}. In the language of the question, we have $S_1 + S_3 = S_2 + S_4$.

We now apply the binomial theorem with $x = 1, y = i$ $(= \sqrt{-1})$:

$$\begin{aligned}
(1+i)^{2016} &= \sum_{k=0}^{\infty} \binom{2016}{k} 1^{n-k} i^k \\
&= \binom{2016}{0} + \binom{2016}{1} i + \binom{2016}{2} i^2 + \binom{2016}{3} i^3 + \ldots \\
&= \binom{2016}{0} + \binom{2016}{1} i - \binom{2016}{2} - \binom{2016}{3} i + \ldots .
\end{aligned}$$

Note that S_1 (respectively S_3) are the positive (respectively negative) terms contributing to the real part of $(1+i)^{2016}$, while S_2 and S_4 make the corresponding contributions to the imaginary part of $(1+i)^{2016}$.

So what are the real and imaginary parts of $(1+i)^{2016}$?

$$(1+i)^{2016} = \left[\sqrt{2}\left(\frac{1+i}{\sqrt{2}}\right)^{2016}\right] = 2^{1008} \cdot [e^{\pi i/4}]^{2016} = 2^{1008} \cdot e^{504\pi i} = 2^{1008}.$$

In particular, this is a positive real number, so that $S_1 > S_3$ and $S_2 = S_4$. Using our previous observation that $S_1 + S_3 = S_2 + S_4 = 2^{2015}$, we get

$$S_1 > S_2 = S_4 = 2^{2014} > S_3.$$

Moreover, the difference beween the largest and smallest S_i is

$$S_1 - S_3 = \mathrm{Re}((1+i)^{2016}) = 2^{1008}.$$

Problem 3 (Pluses and minuses)**.** A sequence of N numbers, each ± 1, has the property that the sum of all N terms is 0 but no block of twenty consecutive terms sums to 0. What is the largest possible value of N?

Answer. 90

Solution. It is not hard to convince yourself that the sequence of length 90 given by

$$+1, -9, +11, -9, +11, -9, +11, -9, +11, -9$$

(meaning one $+1$, followed by nine -1s, etc.) has the desired property. Indeed, in this case the sum of every twenty consecutive terms is $+2$. So it remains to show that there is no suitable sequence of length larger than 90.

Suppose for a contradiction that there is such a sequence of length N, where $N > 90$. For each positive integer $j \leq N - 19$, let S_j denote the sum of the twenty consecutive terms starting with the jth (i.e., the sum of the terms in positions $j, j+1, \ldots, j+19$). By assumption, S_j is nonzero for every j. In fact, S_j is of constant sign. Indeed, suppose S_n and S_m have opposite signs, where $1 \leq n < m \leq N - 19$. It is easy to see that each S_j is even and that each difference

$$S_{j+1} - S_j \in \{-2, 0, 2\}.$$

It follows that the only way S_j can change signs between $j = n$ and $j = m$ is if $S_j = 0$ for some j between n and m. But we have assumed this is not the case!

So without loss of generality, we may assume (reversing the signs of all N terms if necessary) that all S_j are positive, and hence each

$$S_j \geq 2.$$

Now write $N = 20q + r$, where $0 \leq r < 20$. With T denoting the sum of the final r terms in our sequence, we see that the sum of all N terms is

$$S_1 + S_{21} + \cdots + S_{20(q-1)+1} + T.$$

By hypothesis, this sum is zero. Hence,

$$T = -(S_1 + S_{21} + +S_{20(q-1)+1}) \leq -2q. \tag{1}$$

Obviously, $T \geq -r$, and so

$$r \geq 2q. \tag{2}$$

Now the sum of the last 20 terms of the sequence is S_{N-19}, which is ≥ 2. This sum can also be expressed as $T + T'$, where T' is the sum of the $20 - r$ terms preceding the last r. Thus,

$$T \geq 2 - T' \geq 2 - (20 - r) = r - 18. \tag{3}$$

Combining (3) and (1),

$$2q + r \leq 18.$$

Together with (2), this last inequality implies that $4q \leq 18$, and so $q \leq 4$. But $N \geq 92$, and so $q \geq 4$. Thus, $q = 4$, and the last displayed inequality gives $r \leq 18 - 2 \cdot 4 = 10$. But then

$$N = 20q + r = 80 + r \leq 90,$$

contradicting that $N \geq 92$.

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Sources. Problem #1 is inspired by *Euler's Gem: The Polyhedron Formula and the Birth of Topology*, by David S. Richeson (Princeton University Press). Problem #3 is based on work of mathematicians Yair Caro, Adriana Hansberg, and Amanda Montejano.