Sponsored by: UGA Math Department and UGA Math Club

Written test, 25 problems / 90 minutes

October 15, 2011

WITH SOLUTIONS

No calculators are allowed on this test. 10 points for a correct answer, 0 points for an incorrect answer, and 2 points for an answer left blank.

1 Easy Problems

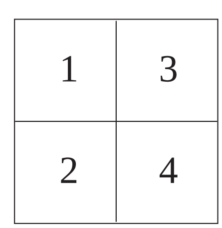
Problem 1. Among families with exactly two children, at least one of which is a boy, what proportion have two boys? (Assume that genders of children are independent and equally likely to be male and female.)

(A) 1/4 (B)$^\heartsuit$ 1/3 (C) 1/2 (D) 2/3 (E) 3/4

Solution. Listing the possible outcomes of the gender of the two children as ordered pairs, ordered by age, there are three equally likely outcomes which involve at least one boy: (B,G), (G,B), (B,B). Since one of those three equally likely outcomes is a success, the answer is 1/3.

Alternative solution: Pick a family at random. If we let X be the event that the family has two boys and Y be the event that the family has at least one boy, then the answer to the question is precisely $P(X|Y)$, read as "the probability of X given Y." The formula for conditional probability is $P(X|Y) = P(X \text{ and } Y)/P(Y)$. In this case, we see that event X guarantees event Y, so $P(X \text{ and } Y) = P(X) = 1/4$. Also, event Y is exactly the complement of the event that the family has two girls, which occurs with probability 1/4, so $P(Y) = 3/4$. Thus, $P(X|Y) = \dfrac{1/4}{3/4} = \dfrac{1}{3}$.

Problem 2. A square piece of paper is subdivided into four congruent squares by a horizontal line and a vertical line, as pictured. The smaller squares are labeled 1, 2, 3, and 4, as shown. You are allowed to fold the paper once over the vertical line—left to right or right to left—and once over the horizontal line—top to bottom or bottom to top—in either order. After



two folds, the smaller squares will be stacked up, and, if we read top to bottom, we get the numbers in some order. How many different permutations of 1, 2, 3, and 4 can we get?

(A) 2 (B) 4 (C)$^\heartsuit$ 8 (D) 12 (E) 24

Solution. It is easy to see that we can get 1 on the top with either 243 or 342 right below it. (No folding can get 1 immediately atop 4.) By symmetry, there are $4 \times 2 = 8$ possible accessible permutations.

Problem 3. Suppose $f(x)$ and $g(x)$ are quadratic polynomials, *all* of whose coefficients are *nonzero*. What is the minimum possible number of *nonzero* coefficients that $f(x)g(x)$ can have?

(A) 1 (B)$^\heartsuit$ 2 (C) 3 (D) 4 (E) 5

Solution. Clearly the coefficient of x^4 and the constant coefficient must be nonzero. However, we can easily arrange for the other three coefficients to vanish: Consider the example

$$(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1) = x^4 + 1\,.$$

(This factoring is essential for calculus students who want to find a function whose derivative is $1/(x^4 + 1)$!)

It is also possible to have an example with integer coefficients:

$$(x^2 + 2x + 2)(x^2 - 2x + 2) = x^4 + 4.$$

Problem 4. A long row of doors, numbered 1, 2, 3, $\ldots$, 100, are all open. Eddie comes along and closes every other door, beginning with the first. Eleanor closes every third door, beginning with the first. Note: Any door that is already closed remains so. This continues until the twenty-fourth person, Eli, closes every twenty-fifth door, beginning with the first. Clearly, the second door is still open. What is the next numbered door that is open?

(A) 27 (B) 28 (C) 29 (D)$^\heartsuit$ 30 (E) 31

Solution. Subtract 1 from each of the door numbers. Then all doors whose number is divisible by 2, 3, $\ldots$ 25 are closed. Clearly, the next prime number 29 is (other than 1) the first one that is open. Returning to the original numbering scheme, the next open door is 30.

Problem 5. Four distinct points A, B, C, and D are chosen at random from 2011 points evenly spaced on a circle. What is the probability that $\overline{AB}$ and $\overline{CD}$ intersect?

(A) 1/4 (B)$^\heartsuit$ 1/3 (C) 680/2011 (D) 1/2 (E) 3/4

Solution. The 2011 is a red (and black) herring. If you pick points A, B, C, D *in order* around the circle, there are $\frac{1}{2}\binom{4}{2} = 3$ pairs of pairs (AB, CD; AC, BD; AD, BC). Only for the pair AC, BD do the two line segments intersect. (For example, in order for $\overline{AB}$ to intersect $\overline{CD}$, C must lie in one of the arcs from A to B and D must lie in the other.) Thus, given 4 random points on the circle, one of the three (equally likely) pairs of line segments intersects.

Problem 6. What is the slope of the line that bisects the angle in the first quadrant formed by the x-axis and the line through the origin with slope 2?

(A) $\dfrac{1}{2}$ (B) 1 (C)$^\heartsuit$ $\dfrac{\sqrt{5}-1}{2}$ (D) 4/3 (E) $\dfrac{\sqrt{5}+1}{2}$

Solution. Say $\tan\theta = 2$. Letting $\phi = \theta/2$, we have $\dfrac{2\tan\phi}{1-\tan^2\phi} = 2$, and so $\tan^2\phi + \tan\phi - 1 = 0$. Thus, $\tan\phi = \dfrac{-1\pm\sqrt{5}}{2}$. We discard the negative solution.

Problem 7. Let $O(n)$ denote the sum of the *odd* digits of n (by this we mean the digits of the numeral that are odd numbers, not the ones in odd decimal places). What is the sum

$$O(1) + O(2) + \cdots + O(100)\,?$$

(A) 251 (B) 276 (C) 500 (D)$^\heartsuit$ 501 (E) None of the above

Solution. Note that $O(1) + O(2) + \cdots + O(9) = 25$, so in each group of ten, the sum of the units digits is 25. The sum of the tens digits gives us another ten groups of 25, and the hundreds digit of 100 gives us an additional 1. Thus, the sum is $(10+10)\cdot 25 + 1 = 501$.

Problem 8. How many monomials $x^a y^b z^c w^d$ are there with the requirement that a, b, c, d are nonnegative integers that sum to 10?

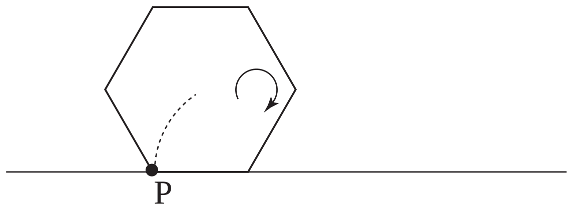
(A) 42 (B) 264 (C)$^\heartsuit$ 286 (D) 1001 (E) None of the above

Solution. Our favorite way of counting here is to put $10 + (4-1)$ dots in a row and mark 3 of them with an X. The number of dots before the first X will be a, the

$$\bullet\ \bullet\ \times\ \bullet\ \bullet\ \bullet\ \bullet\ \times\ \bullet\ \bullet\ \bullet\ \times\ \bullet$$

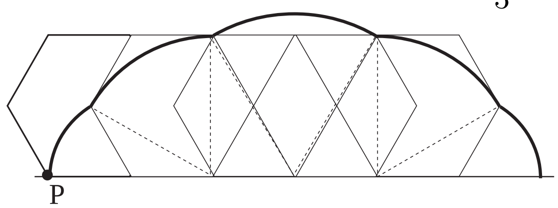
number of dots between the first and second X's will be b, etc. Thus, there are $\binom{13}{3} = 286$ such monomials.

Problem 9. A regular hexagon, as shown, with sidelength 1 "rolls" along a line. What is the length of the path that the vertex P travels as the hexagon rolls through one full cycle?



(A) π (B) 6 (C) $\frac{\pi}{3}(4+\sqrt{3})$ (D)$^{\heartsuit}$ $\frac{2\pi}{3}(2+\sqrt{3})$ (E) $2\pi\big(1+\frac{1}{\sqrt{3}}\big)$

Solution. The path of P consists of five arcs with central angle $\pi/3$, with respective radii 1, $2\sin(\pi/3)=\sqrt{3}$, 2, $\sqrt{3}$ and 1. Thus, the length is $\frac{\pi}{3}\big(4+2\sqrt{3}\big)$.



Christopher Wren found the arclength of a cycloid in 1658 by doing this problem for an n-gon and taking the limit (in the style of Archimedes).

Problem 10. Suppose $3n$ has 56 positive divisors, $6n$ has 70 positive divisors, and $9n$ has M positive divisors. How many different possible values of M are there (that can be achieved by some positive integer n)?

(A) 0 (B) 1 (C) 2 (D)$^{\heartsuit}$ 3 (E) 4

Solution. Let's start with the prime factorization $n = 2^k 3^\ell p_1^{q_1} \dots p_s^{q_s}$ (where p_i are distinct primes larger than 3). For each prime p appearing in this factorization with exponent q, we get a divisor of n by choosing any exponent between 0 and q. Thus, the number of divisors of n is $(k+1)(\ell+1)D$, where $D = (q_1+1)(q_2+1)\dots(q_s+1)$. Similarly, the number of divisors of $3n$ will be $(k+1)(\ell+2)D$, the number of divisors of $6n$ will be $(k+2)(\ell+2)D$, and the number of divisors of $9n$ will be $(k+1)(\ell+3)D$. Thus, if we let $m=(\ell+2)D$, we have $(k+1)m = 56$ and $(k+2)m = 70$, so $\dfrac{k+2}{k+1} = \dfrac{5}{4}$, so $k = 3$. Therefore, $m = (\ell+2)D = 14$, where $\ell \geq 0$ and $D \geq 1$.

There are three possible solutions to this equation, namely $\ell = 0$ and $D = 7$; $\ell = 5$ and $D = 2$; and $\ell = 12$ and $D = 1$. Each of them leads to a different value of M, namely 84, 64, and 60, respectively.

Specific examples for n are $2^3 \cdot 5^6$, $2^3 \cdot 3^5 \cdot 5$, and $2^3 \cdot 3^{12}$, respectively.

2 Medium Problems

Problem 11. Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a real polynomial *all* of whose coefficients are nonnegative integers. Suppose $f(1) = 17$ and $f(20) =$ 496,145. What is a_3?

(A) 0 (B) 1 (C)$^\heartsuit$ 2 (D) 3 (E) 8

Solution. Since all the coefficients of $f(x)$ are nonnegative and $f(1) = 17$, we know that all $a_j \leq 17$. Now, $f(20) = a_n(20^n) + \cdots + a_1(20) + a_0$, so $a_0 \equiv f(20) \equiv 5 \pmod{20}$. Since we know $a_0 \leq 17$, we conclude that $a_0 = 5$. Now subtract $a_0 = 5$ and divide by 20:

$$24{,}807 = \frac{f(20) - 5}{20} = a_n(20)^{n-1} + \cdots + a_2(20) + a_1\,,$$

Mod out by 20 again, and we get $a_1 \equiv 7 \pmod{20}$, and hence $a_1 = 7$. Continuing,

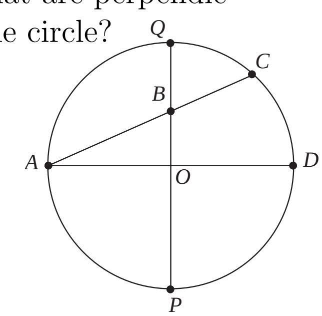
$$1240 = \frac{f(20) - 7(20) - 5}{20^2} = a_n(20)^{n-2} + \cdots + a_3(20) + a_2\,,$$

so $a_2 \equiv 1240 \equiv 0 \pmod{20}$. As before, $a_2 = 0$. Finally,

$$62 = \frac{f(20) - 7(20) - 5}{20^3} = a_n(20)^{n-3} + \cdots + a_4(20) + a_3\,,$$

so $a_3 \equiv 2 \pmod{20}$ and $a_3 = 2$. (For completeness, going one more step, we see that this is a polynomial of degree 4 with $a_4 = 3$, namely $3x^4 + 2x^3 + 7x + 5$.)

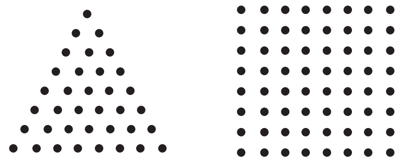
Problem 12. In the figure pictured, $\overline{AD}$ and $\overline{PQ}$ are diameters that are perpendicular to one another. If $AB = 8$ and $BC = 5$, what is the area of the circle?



(A) 30π (B) 36π (C) 45π (D)$^\heartsuit$ 52π (E) 56π

Solution. Although we can do this with the law of sines, it is most instructive to use the "power" of a point: Whenever B is the intersection of $\overline{AC}$ and $\overline{PQ}$, with A, C, P, and Q on a circle, we have $(AB)(BC) = (PB)(BQ)$. [Proof: $\triangle ABP$ is similar to $\triangle QBC$.] If we let $y = OB$ and $r = OP$ (the radius), then we have $y^2 + r^2 = 64$ and $(r+y)(r-y) = 40$, so, adding the two equations, we obtain $r^2 = 52$, so the area of the circle is 52π.

Problem 13. The number 36 has the unusual property that 36 dots can be arranged to form either an equilateral triangle or a square:



If a and b are the next larger integers with this property, what is $\sqrt{ab}$?

(A) 1225 (B) 2011 (C)$^\heartsuit$ 7140 (D) 44100 (E) There are no more numbers with this property.

Solution. In order to arrange x dots in a triangle, we need $x = 1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$ for some natural number n. To arrange them in a square, we need $x = k^2$ for some natural number k. Since n and $n+1$ have no common factors (when $n > 1$), in order to have $\dfrac{n(n+1)}{2} = k^2$, either (if n is even) $n/2$ and $n+1$ must be perfect squares or (if n is odd) n and $(n+1)/2$ must be perfect squares. Either way, we must find two squares so that one differs from the double of the other by 1. Consider the list of perfect squares:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, 121, 144, 169, 196, 225, 256, 289, \ldots$$

We see that $9 = 2 \cdot 4 + 1$, so $n = 8$ gives $x = 36$, which is the example with which we started. Proceeding, we find $49 = 2 \cdot 25 - 1$, so $n = 49$ gives $a = 49 \cdot 50/2 = 1225 = 35^2$ and $289 = 2 \cdot 144 + 1$, so $n = 288$ gives $b = 288 \cdot 289/2 = 144 \cdot 289 = (12 \cdot 17)^2 = 204^2$. Thus, $\sqrt{ab} = 35 \cdot 204 = 7140$.

See also Problem 25 for more information about the general family of equations $x^2 - ky^2 = \pm 1$. This problem is the case $k = 2$, while that problem is $k = 19$.

Problem 14. Among families with exactly two children, at least one of which is a boy born on a Tuesday, what proportion have two boys? (Assume as before that genders of children are independent and equally likely to be male and female, and similarly for day of the week of birth.)

(A) 1/3 (B) 4/7 (C)$^\heartsuit$ 13/27 (D) 1/2 (E) 147/196

Solution. We enumerate the possible outcomes of the gender and day of birth of the two children as ordered pairs, ordered by age, such as $(\mathrm{B}^{\mathrm{Tu}},\mathrm{G}^{\mathrm{F}})$ for a boy born on a Tuesday followed by a girl born on a Friday. We are given that at least one of the coordinates is B^{Tu}. There are 14 outcomes with B^{Tu} in the first coordinate and 14 outcomes with B^{Tu} in the second coordinate. Those two sets overlap in exactly one outcome, leaving a total sample space of $14 + 14 - 1 = 27$ outcomes. There are 7 successes with B^{Tu} in the first coordinate, 7 successes with B^{Tu} in the second coordinate, and again those sets overlap in exactly one outcome, leaving a total of $7 + 7 - 1 = 13$ successes and yielding the answer of 13/27.

Alternative solution: Utilizing the same approach in Problem 1, let X be the event that the family has two boys and let Y be the event that at least one of the children is a boy born on a Tuesday. Note that the event (X and Y) is precisely the intersection of the event X and the event that at least one child is born on a Tuesday (call it T). This alternate formulation is the intersection of two *independent* events, and therefore $P(X \text{ and } Y) = P(X)P(T)$. Of course $P(X) = 1/4$, and T is the complement of the event that neither child was born on a Tuesday, so $P(T) = 1 - (6/7)^2 = 13/49$, and hence $P(X \text{ and } Y) = 13/196$. Also, we see that Y is the complement of the event that neither child is a boy born on a Tuesday, so $P(Y) = 1 - (13/14)^3 = 27/196$. Plugging these values into the conditional probability formula gives the answer 13/27.

Problem 15. Suppose $x^2 + y^2 + 6x - 4y - 12 = 0$. What is largest value that $2x + y$ can have?

(A) $4\sqrt{5} - 4$ (B) 5 (C) 7 (D)$^\heartsuit$ $5\sqrt{5} - 4$ (E) $4\sqrt{5}$

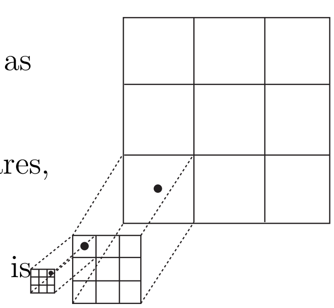
Solution. Note first that (x, y) lies on the circle $(x+3)^2 + (y-2)^2 = 25$, i.e., a circle of radius 5 centered at $(-3, 2)$. Imagine various parallel lines $2x + y = c$ moving from left to right; we'll get the largest possible value on the circle when the line intersects the circle and is as far to the right as it can be. This means it must be tangent. Since the line has slope -2, the line segment from $(-3, 2)$ to the point of contact (x, y) must have slope $1/2$. From the equations

$$\frac{y-2}{x+3} = \frac{1}{2} \qquad \text{and} \qquad (x+3)^2 + (y-2)^2 = 25$$

we now get $\frac{5}{4}(x+3)^2 = 25$, so $x + 3 = 2\sqrt{5}$ and $y - 2 = \sqrt{5}$, and we obtain

$$2x + y = 2(2\sqrt{5} - 3) + (2 + \sqrt{5}) = 5\sqrt{5} - 4\,.$$

Problem 16. Consider the point $P = (a, b)$ with $0 \le a \le 1$, $0 \le b \le 1$. We picture it in the unit square. It has the following properties:



(i) If the square is subdivided into 9 smaller squares, as pictured, then P is in the lower left square.

(ii) If the lower left square is subdivided into 9 smaller squares, P is in the upper left square.

(iii) If that square is subdivided into 9 smaller squares, P is in the upper right square.

(iv) If that square is subdivided into 9 smaller squares, P is in the lower right square.

The subdivision process continues like this, and P continues to be in the lower left, upper left, upper right, lower right squares, forever. What is $a+b$? (The square is oriented so that a increases from left to right and b increases from bottom to top, as usual.)

(A) 1012/9999 (B) 1/3 (C) 10/27 (D)$^\heartsuit$ 2/5 (E) 1/2

Solution. The fact that P is in the lower left square the first time tells us that $0 \le a, b \le 1/3$. In other words, the ternary (base three) "decimal" expansion of a and b each begin with 0. Each successive subdivision tells us the next digit: For example, the second digit of a is 0 and the second digit of b is 2. Indeed, $a = 0.\overline{0022}$ and $b = 0.\overline{0220}$. Next, we add (in base three) and then convert to a standard fraction: Noting there will be no carrying in the repeats, $x = a + b = 0.\overline{1012}$. Now we use the usual technique to turn this into a fraction:

$$3^4 x = 1012_{\text{three}} + x = (27+3+2) + x = 32 + x\,, \qquad \text{so} \quad x = \frac{32}{80} = \frac{2}{5}\,.$$

Problem 17. Notice that the graphs $y = e^x$ and $y = x$ do not intersect in the first quadrant. If a is the smallest positive number so that $y = e^x$ and $y = x^a$ *do* intersect in the first quadrant, which of the following is true?

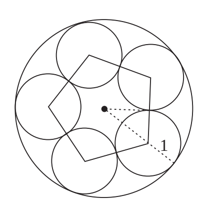
(A) There is no smallest a because $y = e^x$ intersects $y = x^a$ for all $a > 1$. (B) $a = 2$ (C)$^\heartsuit$ $a = e$ (D) $a = \pi$ (E) There is no smallest a because $y = e^x$ fails to intersect $y = x^a$ for all $a > 1$.

Solution. Notice that $y = e^x$ and $y = x^e$ intersect at $x = e$, so this rules out (D) and (E). The graphs intersect at x if and only if $x = a \ln x$, which happens if and only if $\dfrac{\ln x}{x} = \dfrac{1}{a}$. If we graph the function $f(x) = \dfrac{\ln x}{x}$, we see that $f(x) \to -\infty$ as $x \to 0^+$ and $f(x) \to 0$ as $x \to \infty$. It has a global maximum at $x = e$ (as can easily be checked from $f'(x) = (1 - \ln x)/x^2$), and $f(e) = 1/e$. This means that we must have $\dfrac{1}{a} \le \dfrac{1}{e}$, so $a \ge e$.

Problem 18. Five circles of radius 1 are packed tightly so that their centers form the vertices of a regular pentagon. Which of the following is closest to the radius of the smallest circle that will contain them all?

(A) 2.5 (B) 2.6 (C)$^\heartsuit$ 2.7 (D) 2.8 (E) 3

Solution. The radius of the circumscribed circle is $1 + \csc(\pi/5)$. Since $\sin(\pi/5) = \sqrt{\dfrac{5-\sqrt{5}}{8}}$ (see below), it follows that



$$\csc\frac{\pi}{5} = \frac{2\sqrt{2}}{\sqrt{5-\sqrt{5}}} = \frac{\sqrt{2}\sqrt{5+\sqrt{5}}}{\sqrt{5}} = \sqrt{2}\sqrt{1+\frac{1}{\sqrt{5}}}\,.$$

Now, $\sqrt{5} \approx 2.24$, so we can approximate the last by $\sqrt{2}\sqrt{3/2} = \sqrt{3} \approx 1.73$, and so the radius of the circle is approximately 2.73. (In fact, the calculator value is 2.70.)

Let $\theta = \pi/5$. To derive the formula for $\sin\theta$, the most elegant approach is to consider the complex number $\omega = \cos\theta + i\sin\theta$. By DeMoivre's formula, $\omega^5 = (\cos\theta + i\sin\theta)^5 = \cos(5\theta) + i\sin(5\theta) = -1$, so, by the binomial theorem, the imaginary part gives us

$$\sin^5\theta - 10\sin^3\theta\cos^2\theta + 5\sin\theta\cos^4\theta = \sin\theta(\sin^4\theta - 10\sin^2\theta\cos^2\theta + 5\cos^4\theta) = 0\,.$$

Letting $u = \sin^2\theta$, we now have $u^2 - 10(1-u)u + 5(1-u)^2 = 16u^2 - 20u + 5 = 0$, so $u = \dfrac{5 \pm \sqrt{5}}{8}$. Since $0 < \theta < \pi/4$, $\sin^2\theta < 1/2$, so we must have $u = \dfrac{5-\sqrt{5}}{8}$, so $\sin\theta = \sqrt{\dfrac{5-\sqrt{5}}{8}}$, as required.

Problem 19. Five circles of the same radius are packed in a square of unit side length. (That is, the interiors of the circles lie strictly within the square and do not intersect each other. Note that they are not necessarily packed exactly as in the previous problem.) What is the largest possible radius of the circles?

(A) $\frac{2}{4+3\sqrt{2}+\sqrt{6}}$ (B) $\frac{1}{5}$ (C)$^\heartsuit$ $\frac{(\sqrt{2}-1)}{2}$ (D) $\frac{1}{4}$ (E) None of the above

Solution. Consider a putative packing with a radius r. Simply because the circles cannot leave the square, their centers cannot be within r of the boundary, thus confining them to a $(1-2r) \times (1-2r)$ square.

Split this smaller square into four even smaller squares (quarters, so to speak) of the same size, namely $(1/2 - r) \times (1/2 - r)$. By the pigeonhole principle, some two of the five centers must lie in the same quarter. The farthest apart they can be is if the centers lie at opposite corners, so we get that $2r$ (the minimal possible distance between conters) is at most $(1/2 - r)\sqrt{2}$ (the diagonal of this square). It follows that $r \leq \frac{(\sqrt{2}-1)}{2}$.

This radius may be achieved by packing one circle in each corner and one more at the exact center of the square.

Problem 20. Consider a Pascal-like triangle with the numbers 0, 1, 2, 3, ..., going down the edges, as shown:

$$\begin{array}{ccccccccccc}
 & & & & & 0 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 2 & & 2 & & 2 & & & \\
 & & 3 & & 4 & & 4 & & 3 & & \\
 & 4 & & 7 & & 8 & & 7 & & 4 & \\
5 & & 11 & & 15 & & 15 & & 11 & & 5
\end{array}$$

Let $f(n)$ denote the sum of the entries in the row that begins with n. When we divide

$$\frac{f(100)}{f(50)}$$

by 25, what is the remainder?

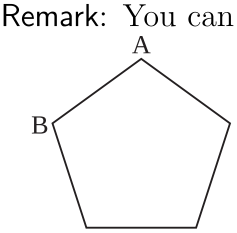
(A)$^\heartsuit$ 0 (B) 2 (C) 3 (D) 24 (E) None of the above

Solution. If we make a "usual" Pascal-type triangle by putting 1's on the outside and let $g(n)$ denote the sum of the entries of the n^{th} row of this modified triangle, then we see that $g(n) = 2g(n-1)$ and $g(0) = 2$. Therefore, $g(n) = 2^{n+1}$ and $f(n) = g(n) - 2 = 2^{n+1} - 2 = 2(2^n - 1)$. Therefore,

$$\frac{f(100)}{f(50)} = \frac{2(2^{100}-1)}{2(2^{50}-1)} = 2^{50} + 1\,.$$

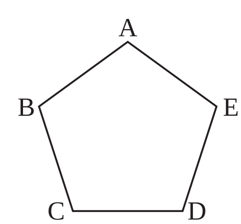
Now, by Fermat, $2^{\phi(25)} \equiv 1 \pmod{25}$, where $\phi(25)$ is the number of integers between 1 and 24, inclusive, relatively prime to 25; it is easy to check that $\phi(25) = 20$, and so $2^{50} \equiv 2^{10} \equiv -1 \pmod{25}$. (Even easier, $2^{10} = 1024 \equiv -1 \pmod{25}$.) Therefore, $2^{50} + 1 \equiv (-1)^5 + 1 \equiv 0 \pmod{25}$.

Problem 21. You start at vertex A of a pentagon. It is equally likely that you move along either of the adjacent edges. You continue, with either of the adjacent edges always having equal probability. You stop as soon as you've visited each vertex. What is the probability that you stop at the adjacent vertex, B? (Remark: You can assume the tour stops after a finite number of steps.)



(A) 1/6 (B) 1/5 (C)$^\heartsuit$ 1/4 (D) 5/18 (E) 1/3

Solution. First, consider the pentagon labeled as shown. Let S be the event that the tour concludes at vertex B. Clearly, for S to occur we must start our tour with the segment AE, so $P(S) = P(S \text{ and} AE) = P(S|AE)P(AE) = \frac{1}{2}P(S|AE)$, so $P(S|AE) = 2P(S)$. Next, having gotten to E, there are three possibilities for the next two moves: EAE, EDE, and EDC. Note that



$$P(S \text{ and } AEDC) = P(S|AEDC)P(AEDC) = 1/8\,;$$

here we use the hypothesis that the tour is known to terminate in a finite number of steps, so, once having arrived at C, it can only conclude at B. (One can prove that with probability 1 the tour does terminate after a finite number of steps.) Next, $P(S \text{ and } AEAE) = P(S|AEAE)P(AEAE) = P(S|AE)P(AEAE) = 2(\frac{1}{8})P(S) = \frac{1}{4}P(S)$. Similarly, $P(S \text{ and } AEDE) = \frac{1}{4}P(S)$. In sum, we have

$$P(S) = P(S \text{ and } AEAE) + P(S \text{ and } AEDE) + P(S \text{ and } AEDC) = \tfrac{1}{2}P(S) + \tfrac{1}{8}\,,$$

from which we conclude that $P(S) = 1/4$. (Amazingly, it is equally likely that the tour terminates at any of the four vertices B, C, D, and E.)

Alternative solution: Consider any vertex P of the pentagon, other than the starting vertex A. In order to you to stop at vertex P, you must first visit *one* of the neighbors of P and then walk all the way around the pentagon and reach P again.

Note that for every vertex P, *one* of its neighbors will be visited at some point. Consider the first such instant. At that point, it no longer particularly matters what vertex P is, nor what path was taken so far. After that point, the only way P will be the final vertex is if the walk walks all the way around the pentagon and hits P "from the other side" before it hits P "from this side". This argument was independent of the non-starting vertex P, hence all of them are equally likely and the answer is $1/4$.

3 Hard Problems

Problem 22. There exist two *distinct* positive integers x and y so that

$$\left(\frac{x}{82}\right)^2 + \left(\frac{y}{82}\right)^2 = 2.$$

What is $|x - y|$?

(A) 18 (B) 24 (C)$^{\heartsuit}$ 36 (D) 62 (E) None of the above

Solution. We seek a rational point (x, y) on the circle

$$x^2 + y^2 = 2.$$

There is a standard method for finding such points, assuming that a single rational point is already known on the circle. Luckily, it's not difficult to provide one, *eg* $P = (-1, -1)$.

The line through point P with rational slope m has equation $y = mx + m - 1$. If we try to intersect this line with the circle, we find that we must have

$$x^2 + (mx + m - 1)^2 = 2,$$

or equivalently

$$(1 + m^2)x^2 + (2m + 2m^2) + (m^2 + 2m - 1).$$

We know one solution to this equation, namely $x = -1$, so by Viète's formula we can compute the other solution is

$$-\frac{2m + 2m^2}{1 + m^2} + 1 = \frac{1 - 2m - m^2}{1 + m^2}.$$

Plugging this back into the equation for y, we see that the overall solution is

$$\left(\frac{1 - 2m - m^2}{1 + m^2}, \frac{1 + 2m - m^2}{1 + m^2}\right).$$

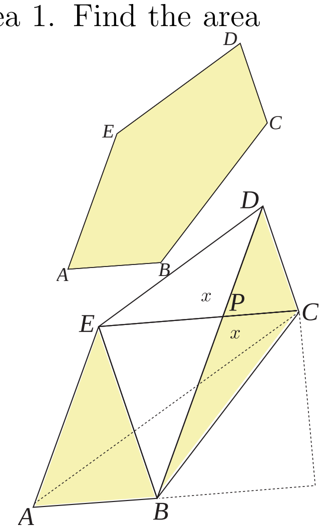
Thus we have found a rational point on the desired circle. In fact it turns out that *all* rational points are obtained this way for some choice of m. (To see this, note that if you connect any rational point with point P, the resulting line has rational slope.)

We seek a solution with denominator 82, so we plug in $m = 9$ and get

$$(-98, -62).$$

The problem wanted positive solutions, so they are 62 and 98, which have a difference of 36.

Problem 23. A pentagon $ABCDE$, as pictured, has the property that each of the triangles $\triangle ABC$, $\triangle BCD$, $\triangle CDE$, $\triangle DEA$, and $\triangle EAB$ has area 1. Find the area of the entire pentagon.



(A) $\dfrac{3+\sqrt{5}}{2}$ (B) 3 (C) $2+\dfrac{\sqrt{5}}{2}$ (D)$^{\heartsuit}$ $\dfrac{5+\sqrt{5}}{2}$ (E) 4

Solution. First we note that since $\triangle ABE$ and $\triangle ABC$ have area 1 and the same base AB, $\overline{CE}$ is parallel to $\overline{AB}$. Similarly, $\overline{BE}$ is parallel to $\overline{CD}$ and $\overline{BD}$ is parallel to $\overline{AE}$. As the diagram indicates, we will know the area of the pentagon once we know the area of $\triangle BED$. First, since $ABPE$ is a parallelogram, the area of $\triangle BEP$ is 1. Denote the area of $\triangle BDE$ by x. Then the area of $\triangle PCD$ is $1-x$, and so the area of $\triangle PBC$ is x as well.

But

$$\frac{\text{area } \triangle EPD}{\text{area } \triangle PCD} = \frac{x}{1-x} = \frac{EP}{PC} = \frac{\text{area } \triangle EPB}{\text{area } \triangle PCB} = \frac{1}{x}.$$

This gives us the equation $x^2 + x - 1 = 0$, so $x = \dfrac{-1+\sqrt{5}}{2}$ (not a huge surprise when pentagons are involved!). At last, the area of pentagon is $3 + x = \dfrac{5+\sqrt{5}}{2}$.

Alternative solution: Do the calculation for a regular pentagon, assuming from the statement of the problem that the answer is independent of the shape of the pentagon. Indeed, it is not hard to show that any pentagon with the given property can be obtained by an affine transformation of a regular pentagon (and affine transformations preserve ratios of areas). As before, given area $\triangle ABE = 1 =$ area $\triangle BCD$, we need to find area $\triangle BDE$. Since each interior angle of a regular pentagon is $108°$, $\angle ABE = \angle CBD = 36°$, and so $\angle EBD = 36°$ as well. From the same reasoning, it follows that $\angle DEC = \angle BEC = 36°$, so $\triangle BEP \cong \triangle BEA$ also has area 1. Therefore, area $\triangle DEP = DP/PB = DE/BE$ (by virtue of the fact that $\overline{EC}$ bisects $\angle BED$). But, by the law of sines, $\dfrac{DE}{BE} = \dfrac{\sin 36°}{\sin 72°} = \dfrac{1}{2\cos 36°} = \dfrac{\sqrt{5}-1}{2}$ (see the solution of Problem 18). So, as before, the area of the pentagon is $3 + \dfrac{\sqrt{5}-1}{2} = \dfrac{5+\sqrt{5}}{2}$.

Problem 24. Andy and Harrison are betting on a seven-game series between the Astronauts and the Hedgehogs. Before each game, they agree to an even-odds bet for some specified amount (possibly zero, possibly much more) on that single game. (The amount of the bet will depend on the score.)

They have structured their bet amounts in advance so that although they are betting on individual games, the sequence of bets is equivalent to a single \$1 bet on the entire series. That is, no matter with what score the Astronauts win (4-0, 4-1, 4-2, or 4-3), Andy will win exactly \$1, and similarly if the Hedgehogs win, Harrison will win exactly \$1.

How much are they betting on the first game? (Assume, if necessary, that money is infinitely divisible.)

(A)$^\heartsuit$ $\$\frac{5}{16}$ (B) $\$\frac{1}{7}$ (C) $\$\frac{1}{4}$ (D) $\$\frac{1}{14}$ (E) \$0

Solution. (We shall drop all \$ signs in this solution.)

This problem can be solved by thinking *backwards*. If the score is 3-3, then the net amount of money exchanged must be 0 and the bet must be 1, because that is the only way of insuring the correct payoffs after this game (which will end the series). If the score is 3-2, then the net amount of money exchanged must be $\frac{1}{2}$ and the bet must be $\frac{1}{2}$, so that if the leading team wins the payoff is 1 while if it loses the net amount of money exchanged is 0. (We know the latter must be true by the previous argument.) We could continue this reasoning or we could summarize it in a table:

	0	1	2	3	4
0	0	$5/16$	$5/8$	$7/8$	$+1$
1	$-5/16$	0	$3/8$	$3/4$	$+1$
2	$-5/8$	$-3/8$	0	$1/2$	$+1$
3	$-7/8$	$-3/4$	$-1/2$	0	$+1$
4	-1	-1	-1	-1	

This table represents the net amount of money that should be exchanged for any given score in the series. The setup of the problem is equivalent to the fact that the payoffs at the very right and the very bottom are $+1$ and -1 respectively, while each internal node is the average of the node below it and to the right of it.

By the way the table is constructed, it is clear that the bet amounts are uniquely determined. So we just read off that the first bet must be for $\frac{5}{16}$.

Problem 25. The Battle of Hastings (October 14, 1066): "The men of Harold stood well together, as their wont was, and formed nineteen squares, with a like [positive] number of men in every square thereof, and woe to the hardy Norman who ventured to enter their redoubts; for a single blow of a Saxon [warrior] would break his lance and cut through his coat of mail.... When Harold threw himself into the fray the Saxons were one mighty square of men, shouting the battle-cries, 'Ut!' 'Olicross!', 'Godemite!'."

What is the smallest number of Saxons (counting Harold) that could have been at the battle?

(A) 15,876 (B)$^\heartsuit$ 28,900 (C) 31,940 (D) 33,856 (E) None of the above

Solution. This problem is asking us to find the smallest positive integers x and y such that $x^2 - 19y^2 = 1$. This is an example of a *Pell equation*. The solutions to the equation $x^2 - 19y^2 = \pm 1$ give us all the numbers of the form $x + y\sqrt{19}$, $x, y \in \mathbb{Z}$, whose multiplicative inverse is again of that form. How many such numbers are there, and how do we find them? Well, there are infinitely many, but they are all of the form $(x_0 + y_0\sqrt{19})^k$ (for some "minimal" x_0 and y_0) as k ranges over the integers.

To find this minimal element, we use the following result: Let n be a square-free positive integer congruent to 2 or 3 mod 4. Let d be the period of the continued fraction expansion for $\sqrt{n}$. Then all numbers we seek are of the form $\pm(p_d + q_d\sqrt{n})^k$ where k is an integer and p_d, q_d are the d^{th} convergents.

Why should we think continued fractions are the way to proceed? Note that the continued fraction of $\sqrt{n}$ is of the form

$$\sqrt{n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{a_4 + \ddots}}}$$

(where all the a_i's are positive integers). The convergents p_i/q_i are the result of truncating this process above at the i^{th} step. Hence, the continued fraction expansion of $\sqrt{19}$ will give good approximations of $\sqrt{19}$. Once we have done that, we will have a good approximation of $p_k^2 = 19q_k^2 + 1$.

The only remaining question would be why the particular convergent listed in the theorem is the way to go. Short answer: Minimality. Longer answer: One can show that if x and y are positive integers such that $x^2 - ny^2 = 1$, then x/y is a convergent of $\sqrt{n}$. Thus, if one is truly skeptical, one can check each convergent until the first solution is obtained.

So consider the continued fraction expansion of $\sqrt{19}$:

i	α_i	$\lfloor \alpha_i \rfloor = a_i$	$\alpha_i - a_i = \frac{1}{\alpha_{i+1}}$	p_i	q_i
-1				0	1
0				1	0
1	$\sqrt{19}$	4	$\sqrt{19} - 4$	4	1
2	$\frac{\sqrt{19}+4}{3}$	2	$\frac{\sqrt{19}-2}{3}$	9	2
3	$\frac{\sqrt{19}+2}{5}$	1	$\frac{\sqrt{19}-3}{5}$	13	3
4	$\frac{\sqrt{19}+3}{2}$	3	$\frac{\sqrt{19}-3}{2}$	48	11
5	$\frac{\sqrt{19}+3}{5}$	1	$\frac{\sqrt{19}-2}{5}$	61	14
6	$\frac{\sqrt{19}+2}{3}$	2	$\frac{\sqrt{19}-4}{3}$	170	39
7	$\sqrt{19} + 4$	8	$\sqrt{19} - 4$	1421	326

Thus, our fundamental unit is $170 + 39\sqrt{19}$. Hence we see that, counting Harold, there were $170^2 = 39^2 \cdot 19 + 1 = 28{,}900$ Saxons at the Battle of Hastings.

Authors. Written by Ted Shifrin, with significant help from Mo Hendon. Alex Rice contributed problems 1 and 14, having learned of them from Colm Mulcahy. Kate Thompson, Derek Ponticelli, and Tyler Kelly contributed problems 25, 11, and 21 respectively. Boris Alexeev contributed problems 19, 22, 24.

Sources. Problem 25 was adapted by Kate Thompson from Neukirch's *Algebraic Number Theory*. Pell's equation and continued fractions are two very deep, very old topics in mathematics. We recommend, among other resources, the following article by Henrik Lenstra in the *Notices of the AMS*:

`http://www.ams.org/notices/200202/fea-lenstra.pdf`

We thank Ed Azoff for his elegant solution of Problem 21. A number of problems were taken from or inspired by AHSME and USAMO.