There are five problems, each worth 20 points. Give complete justification for all assertions by either citing known theorems or giving arguments from first principles.

1. Let \( f : [0, 1] \to \mathbb{R} \) be continuous. Evaluate

\[
\lim_{k \to \infty} \int_0^1 kx^{k-1} f(x) \, dx
\]

2. Let \( \mathcal{B} \) denote the set of all Borel subsets of \( \mathbb{R} \). Let \( \mu : \mathcal{B} \to [0, \infty) \) be a set function with the property that if \( \{E_k\} \) is a countable collection of disjoint sets in \( \mathcal{B} \), then

\[
\mu \left( \bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} \mu(E_k).
\]

(a) Prove that if \( \{F_k\} \) is a sequence of Borel sets for which \( F_k \supset F_{k+1} \) for all \( k \), then

\[
\lim_{k \to \infty} \mu(F_k) = \mu \left( \bigcap_{k=1}^{\infty} F_k \right)
\]

(b) Suppose that for every \( E \in \mathcal{B} \) with Lebesgue measure \( m(E) = 0 \), it follows that \( \mu(E) = 0 \). Prove that for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) so that if \( E \in \mathcal{B} \) with \( m(E) < \delta \), then \( \mu(E) < \varepsilon \).

3. Let \( \{f_k\} \) be any sequence of functions in \( L^2([0, 1]) \) satisfying \( \|f_k\|_2 \leq M \) for all \( k \in \mathbb{N} \). Prove that if \( f_k \to f \) almost everywhere, then

\[
\lim_{k \to \infty} \int_0^1 f_k(x) \, dx = \int_0^1 f(x) \, dx
\]

4. (a) Show that if \( f \in L^4([0, 1]) \), then \( f \in L^2([0, 1]) \) and \( \|f\|_2 \leq \|f\|_4 \).

(b) Does there exist a constant \( C \) so that for all \( f \in L^4([0, 1]) \), \( \|f\|_4 \leq C\|f\|_2 \)? Justify your answer.

(c) For a fixed function \( g \in L^4([0, 1]) \), let \( A \) denote the ratio \( \|g\|_4/\|g\|_2 \). Find a constant \( B \), depending only on \( A \), such that \( \|g\|_2 \leq B\|g\|_1 \).

5. Let \( \mathcal{H} \) be a Hilbert space and \( \{\varphi_k\}_{k=1}^{\infty} \) be a subset \( \mathcal{H} \) with the property that for every \( f \in \mathcal{H} \)

\[
\sum_{k=1}^{\infty} |\langle f, \varphi_k \rangle|^2 = \|f\|^2
\]

where \( \langle f, \varphi_k \rangle \) denotes the inner product of \( f \) and \( \varphi_k \) in \( \mathcal{H} \) and \( \|f\| \) denotes the norm of \( f \) on \( \mathcal{H} \).

(a) Show that for all \( f, g \) in \( \mathcal{H} \),

\[
\langle f, g \rangle = \sum_{k=1}^{\infty} \langle f, \varphi_k \rangle \langle g, \varphi_k \rangle
\]

[Hint: Consider \( \|f + g\|^2 \) and \( \|f + ig\|^2 \)]

(b) Show that for all \( f \) in \( \mathcal{H} \),

\[
\lim_{N \to \infty} \left\| f - \sum_{k=1}^{N} \langle f, \varphi_k \rangle \varphi_k \right\| = 0
\]

[Hint: Use the fact that \( \|h\| = \sup_{\|g\| \leq 1} |\langle h, g \rangle| \) for all \( h \) in \( \mathcal{H} \)]