1. (a) Quote Doob's martingale convergence theorem.
   (b) Define a sequence \( \{X_n, n \geq 0\} \) of random variables as follows: \( X_0 = 0 \); if, for \( n \geq 1 \), \( X_{n-1} = 0 \), define \( X_n = 1, -1, 0 \) with probabilities \( \frac{1}{2n}, \frac{1}{2n}, 1 - \frac{1}{n} \), respectively; if \( X_{n-1} \neq 0 \), define \( X_n = nX_{n-1}, 0 \) with probabilities \( \frac{1}{n}, 1 - \frac{1}{n} \) respectively. Show that (i) \( \{X_n\} \) is a martingale, (ii) \( X_n \rightarrow 0 \) in probability, and (iii) \( X_n \) does not converge to 0 almost surely.

2. (a) Quote (without proof) Kolmogorov's three series theorem.
   (b) If \( \{X_n\} \) is a sequence of independent random variables with zero mean and satisfying
   \[
   \sum_{n=1}^{\infty} E\left[|X_n^21_{|X_n| \leq 1}| + |X_n|1_{|X_n| > 1}\right] < \infty,
   \]
   show that \( \sum_{n=1}^{\infty} X_n \) converges almost surely.

3. (a) Quote a criteria (necessary and sufficient conditions) for the precompactness of a family of probability measures on the space of continuous functions on \([0, 1]\).
   (b) Show that \( X_n \Rightarrow 0 \) if and only if \( X_n \rightarrow 0 \) in probability.

4. (a) If the random variables \( X_1, \ldots, X_n \) are independent with \( E[|X_k|] < \infty, \forall k \), show that
   \[
   E\left\{\prod_{k=1}^{n} X_k\right\} = \prod_{k=1}^{n} E[X_k].
   \]
   (b) Let \( X \) and \( Y \) be independent random variables with distribution functions \( F(x) \) and \( G(y) \) respectively. Show that
   \[
   P(X + Y \leq z) = \int F(z - y) \; dG(y).
   \]

5. (a) Quote (without proof) the First and Second (= direct and converse parts of) Borel-Cantelli Lemmas.
   (b) Let \( \{X_n\} \) be a sequence of independent and identically distributed random variables with zero mean and finite fourth moments. If \( S_n = \sum_{k=1}^{n} X_k \), show that \( S_n \rightarrow 0 \), almost surely.