Instructions: Attempt all problems. Problems 1-4, 6 and 7 are 10 points each. Problems 5 and 8 are 20 points each.

1) a) Let $S$ be a compact space and let $T$ be Hausdorff. Prove that any continuous bijection from $S$ to $T$ is a homeomorphism.

b) Show by examples that both assumptions in a) are necessary.

2) Let $C$ be the "deleted comb space" $C = [0, 1] \times \{0\} \cup \bigcup_{n} \{\frac{1}{n}\} \times [0, 1] \cup \{(0, 1)\} \subset \mathbb{R}^2$

Show that $C$ is connected but not locally connected and not path connected.

3) Show by example that a quotient space of a Hausdorff space need not be Hausdorff.

4) Classify all covering spaces of $\mathbb{RP}^2 \times \mathbb{RP}^2$. Show your reasoning.

5) Let $X = S^1 \vee \mathbb{RP}^2$, the one-point union of $S^1$ and $\mathbb{RP}^2$.

a) Calculate the fundamental group $\pi_1(X)$ (show your work) and describe the universal covering space of $X$.

b) Calculate $H_*(X, \mathbb{Z})$ (show your work).

6) Show that $\text{Free}(x_1, \ldots, x_n)$, the free group on $n$ generators, is isomorphic to a subgroup of $\text{Free}(a, b)$, the free group on two generators.
7) Let $\Sigma_g, g > 0$, be a closed oriented surface of genus $g$. Let $\pi: \Sigma \to \Sigma_g$ be a given connected $k$-fold cover of $\Sigma_g$. Given $g$ and $k$, determine what topological space $\Sigma$ is.

8) a) Suppose $f, g: S^n \to S^n$ are maps with $f(x) \neq g(x)$ for all $x$ in $S^n$. Show that $g$ is homotopic to $A \circ f$ where $A$ is the antipodal map $A(x_1, \ldots, x_{n+1}) = (-x_1, \ldots, -x_{n+1})$.

b) Use the statement of part a) to show that if $f: S^{2n} \to S^{2n}$ is a continuous map, then there exists an $x$ in $S^{2n}$ with $f(x) = x$ or there exists a $y$ in $S^{2n}$ with $f(y) = -y$. 