Algebra Preliminary Exam

Spring 2000

1. How many Sylow subgroups of various orders does a simple group of order 2025 have?
2. For a prime number p, what is the cardinality of $GL_n(\mathbb{F}_p)$? Give two examples of distinct p-Sylow subgroups of $GL_n(\mathbb{F}_p)$.
3. Give an example of a semi-direct product of groups which is not a direct product.
4. The dihedral group D_n can be defined as the group of order $2n$ of symmetries of the regular n-gon in the plane. Write a set of generators and relations for D_n. Find two subgroups H and K of D_4 such that H is normal in K and K is normal in D_4 but H is not normal in D_4.
5. Assume that for $n \geq 5$ the alternating group A_5 is simple. Prove that for $n \geq 5$ the group A_n has no subgroups of order $n!/4$.
6. Prove that any Euclidean domain is a principal ideal domain. Is it true that any principal ideal domain is a unique factorization domain?
7. Give an example of a Euclidean domain which is a free module of finite rank > 1 over $\mathbb{Z}$. Give an example of a unique factorization domain which is not a principal ideal domain.
8. Let K be a splitting field over $\mathbb{Q}$ for $x^6 - 5$.
   (a) Prove that the polynomial $x^6 - 5$ is irreducible over $\mathbb{Q}$.
   (b) What is the degree of K over $\mathbb{Q}$?
   (c) Describe that Galois group of $K/\mathbb{Q}$ as a semi-direct product.
   (d) How many Sylow subgroups (for various primes) does it have?
   (e) Is the extension Galois?
   (f) Describe an intermediate field, say K', between K and $\mathbb{Q}$ such that $K' \neq K$, $K' \neq \mathbb{Q}$ and the extension $K'/\mathbb{Q}$ is Galois.
   (g) What is the Galois group of K' over $\mathbb{Q}$?
9. Given a finite field $\mathbb{F}_p$ of cardinality q and characteristic p, prove that it is equal to the set of roots and the splitting field of a polynomial over $\mathbb{F}_p$. Which polynomial?