Topology Qualifying Exam
January 2003

1. A space $X$ is *sequentially compact* if every sequence in $X$ has a convergent subsequence. Prove that a compact metric space $X$ is sequentially compact.

2. Prove that if $X$ is Hausdorff and $Y$ is a retract of $X$ then $Y$ is closed in $X$.

3. Describe the universal covering space of the Klein bottle and the group of deck transformations of this covering space.

4. Find all the connected 2-sheeted covering spaces of the figure 8 (the wedge product of two circles). Use this classification to find all the index 2 subgroups of the free group on two generators.

5. Find all the compact orientable surfaces which are covering spaces of a compact orientable surface of genus 3. (Hint: If $X$ is an $n$-sheeted covering space of the finite CW complex $Y$, then the Euler characteristic of $X$ is $n$ times the Euler characteristic of $Y$.)

6. Use the Mayer-Vietoris sequence to compute the homology of the Klein bottle by writing the Klein bottle as the union of two Möbius strips along their boundaries.

7. Prove that for all $n > 0$ the unit sphere $S^{n-1}$ in Euclidean $n$-space $\mathbb{R}^n$ is not a retract of $\mathbb{R}^n$.

8. State the Lefschetz Fixed Point Theorem. Prove that if $f$ is a continuous map from complex projective $n$-space to itself, $n \geq 0$, and $f$ is homotopic to the identity map, then $f$ has a fixed point. If you know the homology groups of $\mathbb{C}P^n$, you do not need to rederive them.