Numerical Analysis Preliminary Examination
Spring, 2011

NAME _________________________ SCORE _______________________

Instruction: Do all problems and show all your work.

[1] (10pts) Consider the Steffensen method for nonlinear equation \( f(x) = 0 \):

\[
x_{n+1} = x_n - \frac{f(x_n)^2}{f(x_n + f(x_n)) - f(x_n)}.
\]

Show that this is quadratically convergent under suitable hypotheses. Please state the hypotheses and give your proof.

[2] (10pts) Let \( x_k \) and \( x_{k+1} \) be two successive iterates when Newton’s method is applied to find the zeros of a polynomial \( p \) of degree \( n \). Show that there is a zero of \( p \) within distance \( n|x_k - x_{k+1}| \) of \( x_k \).

[3] (10pts) Suppose that a matrix \( A \) is diagonally dominant. Show that the Gauss-Jacobi’s method for \( Ax = b \) converges.

[4] (10pts) Suppose that \( A \) is weakly diagonally dominant and is irreducible. Show that the Gauss-Jacobi’s method for \( Ax = b \) also converges.

[5] (10pts) Find a Householder’s transformation to convert the following vector \( \mathbf{v} \) into \( [0, 0, 0, \alpha]^T \) with \( \alpha \) being the norm of the vector \( \mathbf{v} \):

\[
\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\]

[6] (10pts) Explain how one can find an orthonormal matrix \( Q \) and a lower triangular matrix \( L \) for a given square matrix \( A \) such that \( A \) can be factored into \( A = QL \).

[7] (10pts) Define the QR iterative method for numerical solution of eigenvalues of symmetric matrix \( A \). Explain each step.

[8] (10pts) Define the least square data fitting problem and explain how to use the SVD method to solve the least square data fitting problem.

[9] (10pts) Let \( a = x_0 < x_1 < \cdots < x_n < x_{n+1} = b \) be a partition of \([a, b]\). For \( f \in C[a, b] \), let \( S_f \) be the \( C^2 \) natural cubic interpolatory spline of \( f \), i.e.,

\[
S_f(x_i) = f(x_i), \quad i = 0, 1, \cdots, n + 1, \quad S'_f(a) = 0 = S''_f(b),
\]

Suppose that \( f \in C^2[a, b] \). Show that

\[
\int_a^b |S''_f(x)|^2 dx \leq \int_a^b |f''(x)|^2 dx.
\]

[10] (10pts) Let \( B^n_i(x) \) be B-spline of order \( n \) over knots \( x_i, x_{i+1}, \cdots, x_{i+n} \). Show that \( B_i(x) \geq 0 \) and

\[
\sum_i B^n_i(x) = 1.
\]