1. (a) What does it mean to say a topological space is *compact*?
   (b) Show that a product space $X \times Y$ is compact if and only if both $X$ and $Y$ are compact.

2. (a) What does it mean to say a topological space is *connected*?
   (b) What does it mean to say a topological space is *path-connected*?
   (c) Prove that if $X$ is path-connected, then $X$ is connected.
   (d) Either prove or give a counterexample to the converse of part (c).

3. Fix an integer $n$. Let $X$ be the surface obtained from the cylinder $S^1 \times [0,1]$ by identifying the point $(z,0)$ with the point $(z^n,1)$ for all $z \in S^1$. Find the fundamental group of $X$.

4. Calculate the homology groups of $S^2 \times S^2$.

5. (a) What is meant by the *degree* of a continuous map $f : S^n \to S^n$?
   (b) Let $f : S^n \to S^n$ be a continuous map with no fixed points. Prove that $f$ has degree $(-1)^{n+1}$.

6. Suppose that $X$ has universal cover $p : \tilde{X} \to X$ and let $A$ be a subspace of $X$ with $p(\tilde{a}) = a \in A$. Show that there is a group isomorphism
   $$\ker(\pi_1(A,a) \xrightarrow{i_*} \pi_1(X,a)) \cong \pi_1(p^{-1}(A),\tilde{a})$$
   where $i : A \to X$ is the inclusion map.

7. (a) State the classification theorem for compact connected surfaces (with or without boundary).
   (b) Show that any compact connected surface with a nonempty boundary is homotopy equivalent to a wedge of circles. (Hint: you may assume that any compact connected surface without boundary is given by identifying edges of a polygon in pairs.)
   (c) For each surface in your classification, say how many circles are needed in the wedge from part (b). (Hint: you should be able to do this even if you have not done part (b).)

8. For which compact connected surfaces $\Sigma$ (with or without boundary) does there exist a continuous map $f : \Sigma \to \Sigma$ that is homotopic to the identity map and has no fixed points? Explain your answer fully.