# Topology Qualifying Exam

## August, 2013

Justify all the calculations and state the theorems you use in your answers.

1. Prove that the product of two compact spaces is compact.
2. Give an example of a topological space that is connected but not path connected. Make sure you prove that your example satisfies the property.
3. Prove that the Euler characteristic of a compact surface with boundary which has $k$ boundary components is $\leq 2-k$.
4. Give the definition of a covering space $\hat{X}$ (and covering map $p : \hat{X} \to X$) for a topological space $X$. Prove that if $X$ is simply connected, and $(\hat{X}, p)$ is a covering space for $X$, then $p$ is a homeomorphism. Make your proof as elementary as possible.
5. Use covering spaces to show that any free group is a subgroup of $F_2$, the free group on 2 generators.
6. Let $X$ be a space obtained by attaching two 2-cells to the torus $S^1 \times S^1$, one along a simple closed curve $1 \times S^1$ and the other along $S^1 \times 1$. Calculate the fundamental group and homology groups of $X$.
7. Use Mayer-Vietoris sequence to calculate the homology groups of $S^n$.
8. Compute the homology groups $H_i(\mathbb{R}P_n/\mathbb{R}P_m; \mathbb{Z})$ of the quotient space $\mathbb{R}P_n/\mathbb{R}P_m$, for $m < n$. Use cellular homology, using the standard CW structure on $\mathbb{R}P_n$ with $\mathbb{R}P_m$ as its m-skeleton.
9. Let $S$ be an oriented surface and let $f : S \to S$ be a continuous map homotopic to the identity that has no fixed points. Find all the possible values for the genus of $S$. Prove your answer is true.