The following nine questions have equal weight (11 points each) - answer all of them.

**Q1:** For any integer $n \geq 2$ let $X_n$ denote the space formed by attaching a 2-cell to the circle $S^1$ via the attaching map $a_n : S^1 \to S^1$ defined by $e^{i\theta} = e^{in\theta}$. (In other words, $X_n$ is the quotient of the disjoint union of the disk $D^2$ and the circle $S^1$ by the relation which identifies a point $e^{i\theta}$ on the boundary of $D^2$ with $e^{in\theta} \in S^1$.)

(i): Compute the fundamental group and the homology of $X_n$. (6 points)
(ii): Exactly one of the $X_n$ (for $n \geq 2$) is homeomorphic to a surface. Identify, with proof, both this value of $n$ and the surface that $X_n$ is homeomorphic to (including a description of the homeomorphism). (5 points)

**Q2:**
(i): State what it means for a covering space to be normal (sometimes this condition is instead called regular). (4 points)
(ii): Let $\Theta$ be the topological space formed as the union of a circle and a diameter of the circle (so this space looks exactly like the letter $\Theta$). Give an example of a covering space of $\Theta$ which is not normal. (7 points)

**Q3:** Give a self-contained proof that the 0th singular homology $H_0(X)$ is isomorphic to $\mathbb{Z}$ for every path-connected space $X$.

**Q4:**
(i): Prove that for every continuous map $f : S^2 \to S^2$ there is some $x$ such that either $f(x) = x$ or $f(x) = -x$. (Hint: Where $A : S^2 \to S^2$ is the antipodal map, you are being asked to prove that either $f$ or $A \circ f$ has a fixed point.) (7 points)
(ii): Exhibit a continuous map $f : S^3 \to S^3$ such that for every $x \in S^3$, $f(x)$ is equal to neither $x$ nor $-x$. (Hint: It might help to first think about how you could do this for a map from $S^1$ to $S^1$.) (4 points)

**Q5:** Suppose that $U$ and $V$ are open subsets of a space $X$, with $X = U \cup V$. Find, with proof, a general formula relating the Euler characteristics of $X$, $U$, $V$, and $U \cap V$.

**Q6:** Prove that any finite tree is contractible, where a tree is a connected graph that contains no closed edge paths.

**Q7:** Let $X$ be a topological space.

(i): State what it means for $X$ to be Hausdorff. (3 points)
(ii): Prove that every compact subset of a Hausdorff space is closed. (8 points)

**Q8:** Let $X$ be a topological space.

(i): State what it means for $X$ to be compact (3 points).
(ii): Let

$$X = \{0\} \cup \{1/n \mid n \in \mathbb{Z}_+\}.$$  

Is $X$ compact? (4 points)
(iii): Let

$$X = (0,1].$$

Is $X$ compact? (4 points)

**Q9:** Let $X$ be a path connected topological space.

(i): Prove that $X$ is connected. (5 points)
(ii): Is the converse true? Prove or give a counter example. (6 points)