Topology Qualifying Exam: Spring 2013

1. Recall that a topological space X is said to be connected if there does not exist a pair U, V of disjoint nonempty open subsets of X whose union is X.
   i. Prove that X is connected if and only if the only subsets of X that are both open and close are X and the empty set.
   ii. Suppose that X is connected and let $f : X \to \mathbb{R}$ be a continuous map. If a and b are two points of X and if r is a point of $\mathbb{R}$ lying between $f(a)$ and $f(b)$, show that there exists a point c of X such that $f(c) = r$.

2. i. Let X be a topological space. Give the definition of the statement "X is compact."
   ii. Prove that every compact subspace of a Hausdorff space is closed.

3. i. Let A be a subspace of a topological space X. Give the definition of the statement "A is a deformation retract of X."
   ii. Consider X_1 the "planar figure eight," and $X_2 = \mathbb{S}^1 \cup (\{0\} \times [-1, 1])$ (the "theta space"). Show that X_1 and X_2 have isomorphic fund groups.
   iii. Prove that the fund group of X_2 is a free group on two generators.

4. i. Let S_1 and S_2 be disjoint surfaces. Define their connected sum, $S_1 \# S_2$.
   ii. Compute the fund group of the connected sum of the projective plane and the two torus.

5. i. State the classification theorem for compact surfaces.
   ii. What surface is represented by the 6-gon with the edges identified in pairs according to the symbol
   $$xyzxy^{-1}z^{-1}?$$

6. Show that any continuous map $f : RP^2 \to S^1 \times S^1$ is necessarily null-homotopic.

7. Let A be a 3 x 3 orthogonal matrix. Let $f : S^2 \to S^2$ be the restriction of A to S^2. Show that there is a pair of antipodal points mapped by f to the same pair of antipodal points.

8. Does there exist a map of degree 2013 from S^2 to S^2?