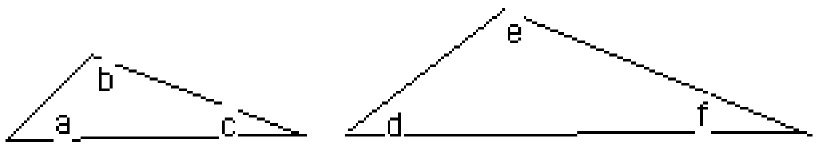
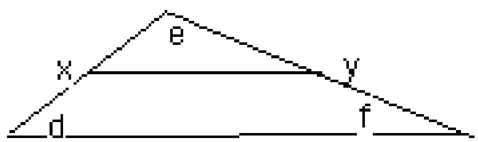
5200: Similarity of figures.

We understand pretty well figures with the same shape and size. Next we study figures with the same shape but different sizes, called similar figures. The most important ones are triangles.

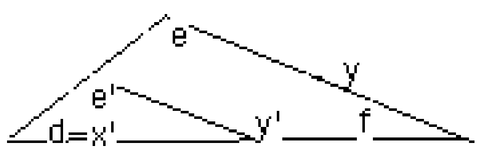
Define: Two triangles to be similar if they have the same angles. I.e. triangles <abc> and <def> are similar (in that order) if angles a and d are equal, and also angles b and e, and angles c and f.



For example, this happens if the sides of <abc> are all twice as long as the corresponding sides of <def>, as we will prove next. Recall we have proved that a line which is parallel to the base of triangle and bisects one side, also bisects the other side it meets.



For instance, here if x is the midpoint of side de, and xy is parallel to df, then y is the midpoint of ef. Here by the Z principle, all angles of triangle <exy> are equal to those of triangle <edf>. Moreover, side ex is half as long as side ed. Hence if we slide triangle exy down side ed, until x rests at point d, we have this picture, where triangle <e'x'y'> is congruent to triangle <exy>.



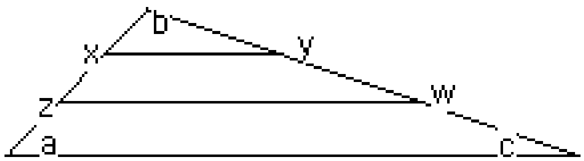
Since angle <x'e'y' equals angle e, side e'y' is parallel to side ef. Thus since e' is the midpoint of side de, then y' is the midpoint of side df. That is, side x'y' is half as long as side df. Hence all three sides of triangle <exy> are half as long as the corresponding sides of triangle <edf>.

Now begin from triangle <abc> and construct triangle <def> with sides double those of <abc>. Then by our construction there is a triangle <exy> with the same angles as <edf> and sides half the length of those of <edf>. Since the sides of <exy> are then the same as those of <abc> it follows that <exy> and <abc> are congruent by SSS, so have the same angles. Hence triangle <abc> has the same angles as does its double, triangle <def>.

We can do the same for triples of triangles, or any multiple, using exactly the same argument, but we repeat it once more for emphasis.

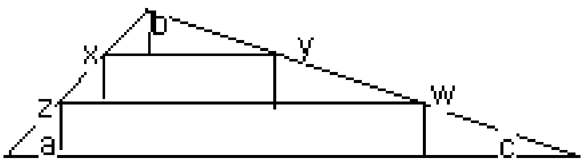
Lemma: Lines parallel to the base of a triangle which trisect one side, also trisect the other side they meet.

proof:



Assume xy and zw are parallel to side ac, and points x,z trisect side ab, and points y,w trisect side bc.

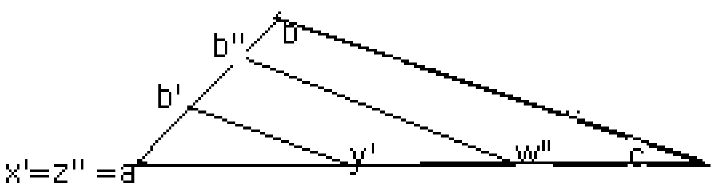
proof: Drop perpendiculars again forming three congruent right triangles on each side,



hence the three hypotenuses by, yw, and wc, are all the same length.

QED.

Note the three triangles <bxy>, <bzw>, and <bac> are all similar. Hence if we slide the first two of these triangles down side ab, until points x and z come to lie on point a, sides b'y' and b''w'' are parallel to side bc, b',b'' trisect ab, and the same argument shows points y', w'', trisect side ac.



Reasoning backwards via SSS, proves the triple of a triangle is similar to the original triangle.

Proposition: For every integer n>0, lines parallel to one side of a triangle, and dividing that side into n equal segments, also divide the other side they meet into n equal segments.

proof: exercise.

Cor: Given a triangle <abc>, any triangle <def> whose sides are n times as long as those of <abc>, is similar to <abc>. Also if <def> is a triangle, any triangle <abc> whose sides are 1/n times as long as those of <def> is similar to <def>.

Now we have enough to prove the basic fact about similar triangles.

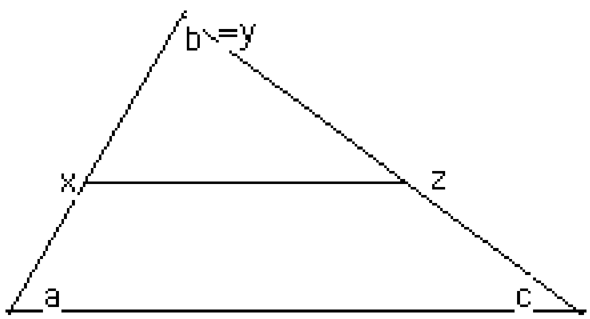
Theorem: Two triangles <abc> and <def> are similar (in that order) if and only if there is a real number r such that the sides of <def> are r times as long as the corresponding ones of <abc>. I.e. angle a equals angle d, and angle b equals angle e, and angle c equals angle f, if and only if for some real number r>0, $|ab/|de| = |bc|/|ef| = |ca|/|fd| = r$.

Remark: This is actually a "triviality" at this point, but a tedious one. I.e. it uses very little geometry, but some tedious reasoning about real numbers. I.e., the theorem is true by the previous proposition for the case of r rational. Since all real numbers can be approximated by rationals as closely as desired, it follows for all reals as well. This is due apparently to Eudoxus.

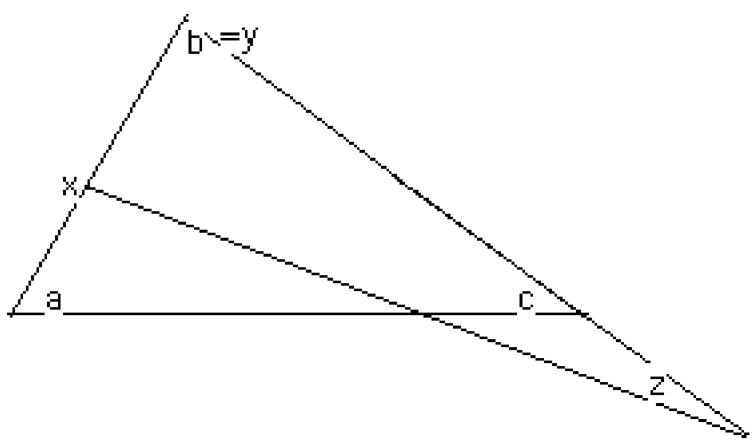
The proof looks horrible and long in Millman / Parker, so we will see if I can do any better. Peeking in Euclid might help, but I will understand it better if I do it myself.

The whole point: Triangles have proportional sides iff they have equal angles.

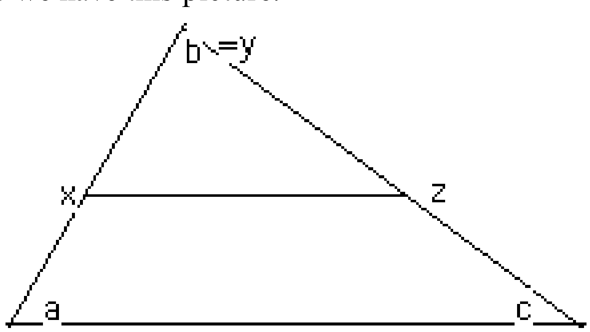
Theorem 1: If triangles <abc> and <xyz> have corresponding angles equal: <a = <x, < b = <y, and <c = <z, then the ratio of the lengths of any two sides is a constant, e.g. |ab|/|xy| = |bc|/|yz|.
proof: If |xy| < |ab|, place angle y inside angle b, so point y is at point b and side xy is along side ab.



We claim point z is between b and c. Otherwise, the picture looks like this:



But then the exterior angle thm. says angles acb and xzy are not equal.
So we have this picture:



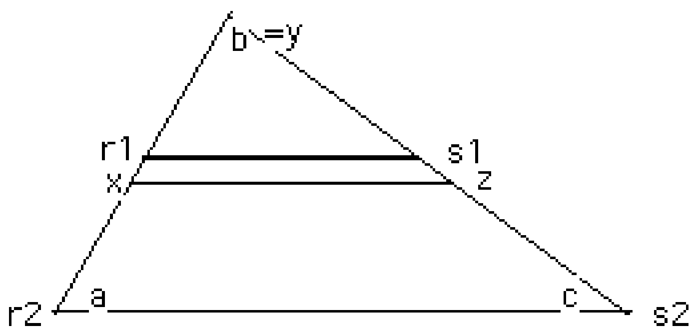
and now since angles <xzb and <acb are equal, line xz is parallel to line ac.

Suppose by accident that r = |bx|/|ba| = k/n is a rational number. Then if we subdivide side ab into exactly n equal pieces, x will lie on the kth subdivision point. And if we draw lines parallel to the base ac, through each subdividing point, these lines will meet side bc and will subdivide it into n equal segments, and z will lie on the kth one, so |bz|/|bc| will also equal r = k/n.

Thus we only need deal with the case where the ratio |bx|/|ba| is irrational. The same argument shows that every rational number smaller than |bx|/|ba| is also smaller than |bz|/|bc|. Hence |bx|/|ba| = |bz|/|bc|. We do some of the slightly tedious details of this aregument.

We will show for every integer n >0, that the absolute value of the difference |bx|/|ba| - |bz|/|bc| is < 1/n. Since no positive number can be less than 1/n for every n, it will follow that this difference is zero, i.e. that |bx|/|ba| = |bz|/|bc|. We illustrate it for n = 2, 3,...

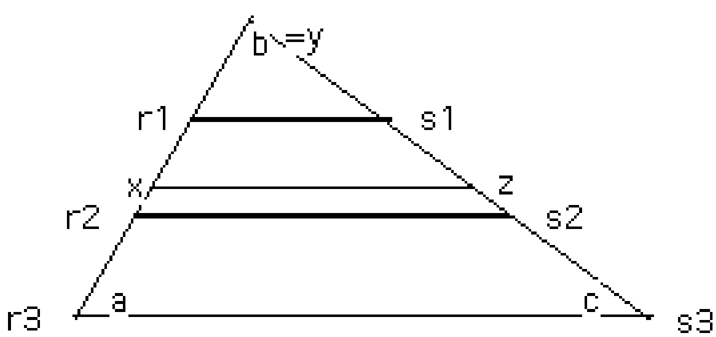
Suppose we subdivide the sides ab and bc in half by a line (r1s1) parallel to the base ac. Then that line is also parallel to line xz, so we have some picture like this:



It follows that the ratio |bx|/|ba| is greater than $|br_1|/|br_2|$ = 1/2, but less than 1, and the same holds for the ratio |bz|/|bc|. I.e. both ratios are between 1/2 and 1, so they are closer together than 1/2. Hence the absolute value of the difference |bx|/|ba| - |bz|/|bc| is less than 1/2.

Now the point x could have been on the other side of the midpoint r1, but it does not matter, since then the ratios |bx|/|ba| and |bz|/|bc| are both between 0 and 1/2, hence still closer together than 1/2. All that matters is that since the parallel lines cannot cross each other, the points x and z are in the same half of sides ab and bc.

Now do it again for n = 3.

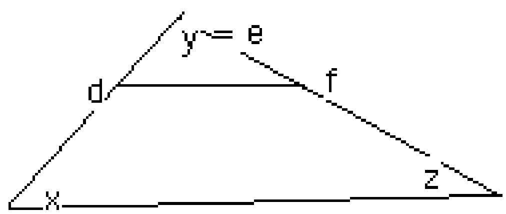


This time if we have x between r1 and r2, then z is also between s1 and s2. So the ratios |bx|/|ba| and |bz|/|bc| are both between 1/3 and 2/3, hence closer together than 1/3. Again it does not matter which dividing point they are between, just that they are between the ones with the same index, i.e. if x is between r_i and r_{i+1}, then z is also between s_i and s_{i+1}, since the parallel lines cannot cross each other.

Hence we have proved that the ratios |bx|/|ba| and |bz|/|bc| are closer together than 1/3, so the absolute value of the difference |bx|/|ba| - |bz|/|bc| is less 1/3.

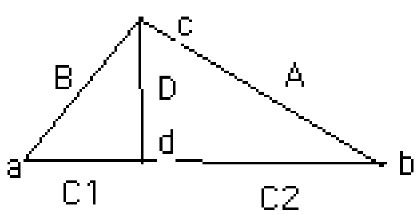
Since we can subdivide the sides into any number n of equal pieces, we can show in the same way that the difference |bx|/|ba| - |bz|/|bc| has absolute value less than 1/n for every n>0. This implies it is zero, and hence the ratios |bx|/|ba| and |bz|/|bc| are equal. **QED.**
This argument applies to any two sides.

Theorem 2: Conversely, if two triangles <abc> and <xyz> have proportional sides, i.e. if the ratios |ab|/|xy|, |bc|/|yz|, and |ca|/|zx| are all equal real numbers, then the angles of the triangles are equal in the same corresponding order, i.e. <a = <x, <b = <y, and <c = <z.
proof: This is reasoning backwards as before, via SSS. I.e. given the two triangles, if side |ab| < side |xy|, draw a triangle <def> inside triangle <xyz> similar to <xyz>, and with point e located at point y, and side de lying along side xy, with side |de| equal to side |ab|.



Then since triangles <def> and <xyz> are similar, these ratios are equal |ef|/|yz| = |ed|/|yx| = |df|/|xz|. Since by hypothesis, side |de| = side |ab|, we have also side |ef| = side |bc|, and side |yz| = side |ac|. Hence by SSS, triangles <def> and <abc> are congruent, hence have same corresponding angles, i.e. <abc> is indeed similar to <xyz>. **QED.**

Cor: (Numerical Pythagoras): If <abc> is a right triangle with right angle at c, and sides opposite a,b,c of lengths A,B,C, then $A^2+B^2 = C^2$.
proof: draw in a perpendicular from c to ab meeting it at d.



Here $C = C_1+C_2$.

Then triangles <adc> and <cdb> are both similar to <acb>. Hence their sides are proportional, viz. $A/C_2 = C/A$, and $B/C_1 = C/B$. Thus $A^2 = CC_2$, and $B^2 = CC_1$, hence $A^2+B^2 = CC_1+CC_2 = C(C_1+C_2) = C^2$. **QED.**

Cor: Conversely if $A^2+B^2=C^2$ in triangle <abc>, then angle c is 90.
proof: There is a triangle <xyz> with sides X = A, Y = B, and angle <Z = 90. Then $C^2 = A^2+B^2 = X^2+Y^2 = Z^2$, so C =Z, the triangles are congruent by SSS, hence <c = <z = 90. **QED.**