Euclid's treatment of Pythagoras
[Remember we are assuming the parallel postulate EFP now.]

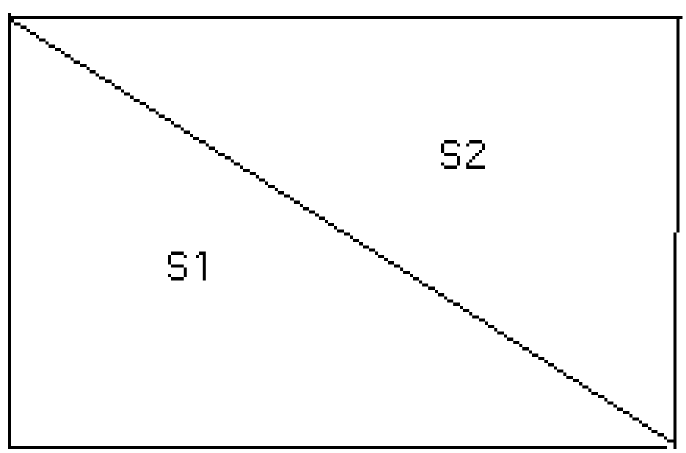
Euclid did not use real numbers to measure lengths or areas. Thus his statement of Pythagoras' theorem was one that did not mention numerical areas, in particular he did not say that a^2 + b^2 = c^2 as we do. He had to say the square on the hypotenuse could be obtained from the two squares on the legs, by some process which would preserve area, if we had a notion of area. The simplest such notion is having "congruent dissections". I.e. two figures have congruent dissections if they can be constructed from congruent pieces, possibly in different arrangements.

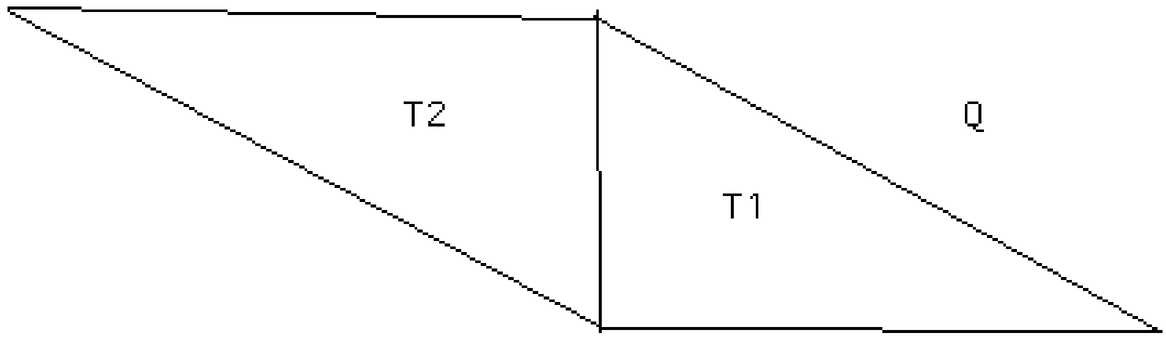
More precisely, define a "**figure**" as a finite non overlapping union of triangles, where "non overlapping" means no point interior to one triangle is also interior to another triangle. To save words we will say "sum" for "non overlapping union". E.g., a parallelogram is the sum of two triangles having a diagonal of the parallelogram as a common side.

Define two figures P,Q to have "**congruent dissections**", if there exist triangles $T_1,\ldots,T_n$ and $S_1,\ldots,S_n$, such that for every i, S_i and T_i are congruent, and P is the non overlapping union of $S_1,\ldots,S_n$, while Q is the non overlapping union of $T_1,\ldots,T_n$.

For example if we take a rectangle P, and cut it into two triangles by a diagonal, we can reassemble those two triangles to form a parallelogram Q. Then P and Q have congruent dissections, since P is a sum of two triangle S_1, S_2, and Q is a sum of two triangles T_1,T_2, where S_1 and T_1 are congruent, and S_2 and T_2 are congruent. In fact here all 4 triangles may be taken as congruent. [compare picture on next page.]

However, this notion is not the easiest one in the world to work with, as it can be challenging to show that two figures actually do have congruent dissections.



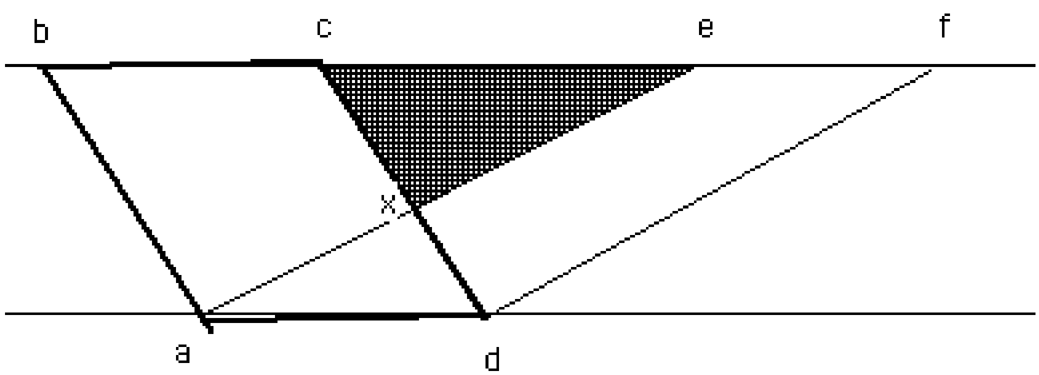


An easier notion, apparently weaker, but actually always equivalent (after a lot of work), is the following one, of having "equal content".

Define P,Q to have "**equal content**", if there exist figures R,S, so that R and S have congruent dissections, and such that the sums P+R, and Q+S, also have congruent dissections. (It is not obvious, but it is true, than then P and Q themselves already have congruent dissections.)

We will assume without proof, at least until later, **that both notions, having congruent dissections, and having equal content, are transitive, and hence define equivalence relations on figures**. (See Hartshorne, pages 199, 201.) It can also be shown that every polygon is a figure, i.e. can be expressed as a sum of triangles (see e.g. Einar Hille, Analytic Function Theory, app. B, vol. 1.)

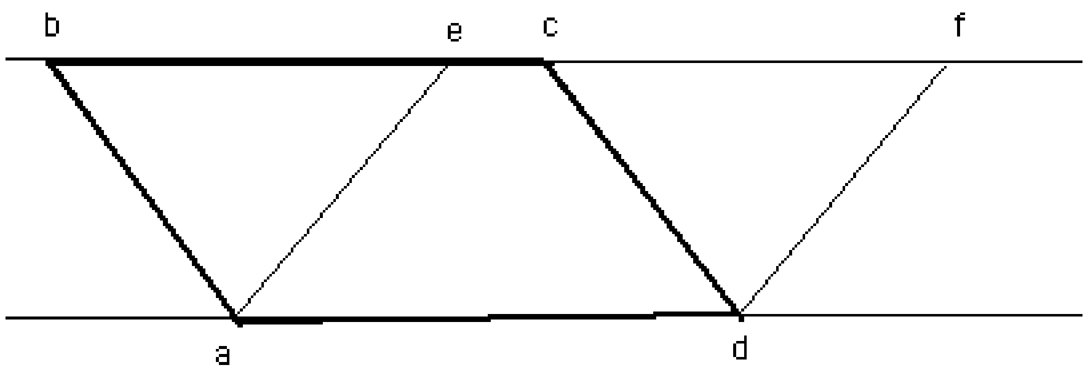
Prop. 35 (Euclid) Parallelograms which are on the same base and "in the same parallels" [i.e. have collinear sides opposite the common base], are “equal”, [i.e. have equal content].
proof: Euclid's picture shows only one case, where the sides opposite the common base are disjoint, as below. (This picture is reproduced in Hartshorne, p. 198. ex. 22.1.2, apparently without commenting that it is not the general case.)



We want to show parallelograms [abcd], and [aefd], have equal content. We claim that adding the same triangle, <cxe>, to both of them, makes the resulting figures have congruent dissections.

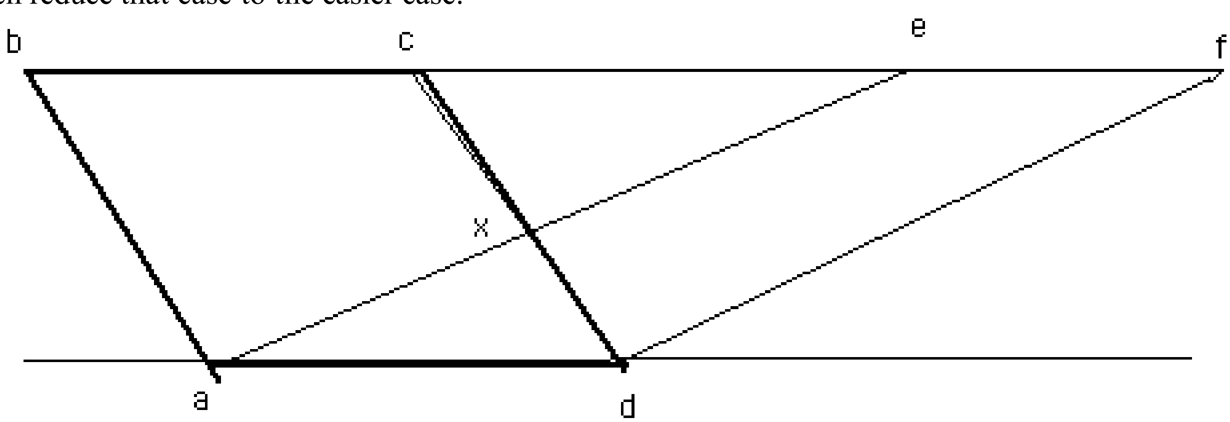
I.e. the sum of [abcd] and <cxe> can be dissected into triangles <bae> + <axd>. While the sum of [aefd] and <cxe>, can be dissected into <cdf> + <axd>. Since triangles <bae> and <cdf> are congruent, indeed figures [abcd] + <cxe>, and [aefd] + <cxe>, have congruent dissections. Thus parallelograms [abcd] and [aefd] have equal content as claimed.

This proof works as long as point e, is on or to the right of point c. But we could also have the following picture:

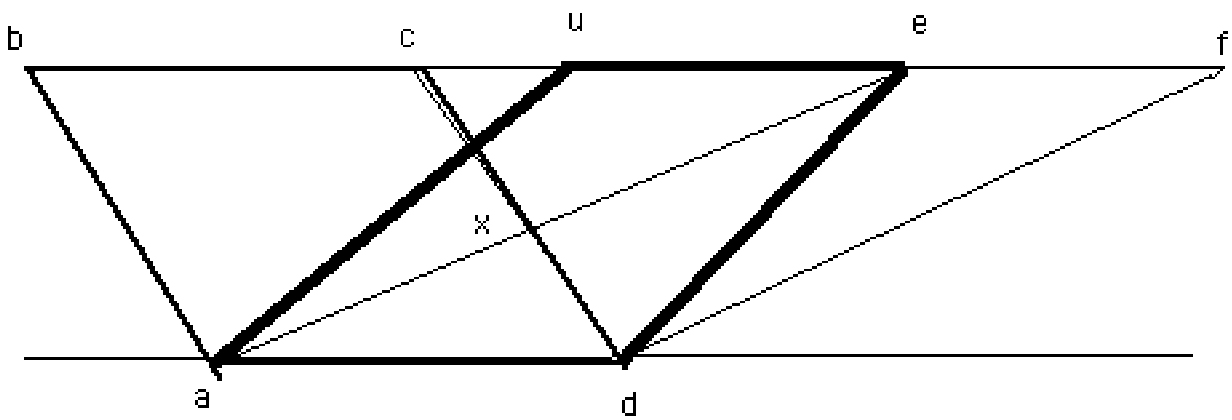


Now there is no point x as before. But this case is easier than before, since triangles <abe> and <dcf> are congruent, so even without adding anything, parallelograms [abcd] and [aefd], have congruent dissections. So in this case we have proved something stronger than equal content for the two parallelograms.

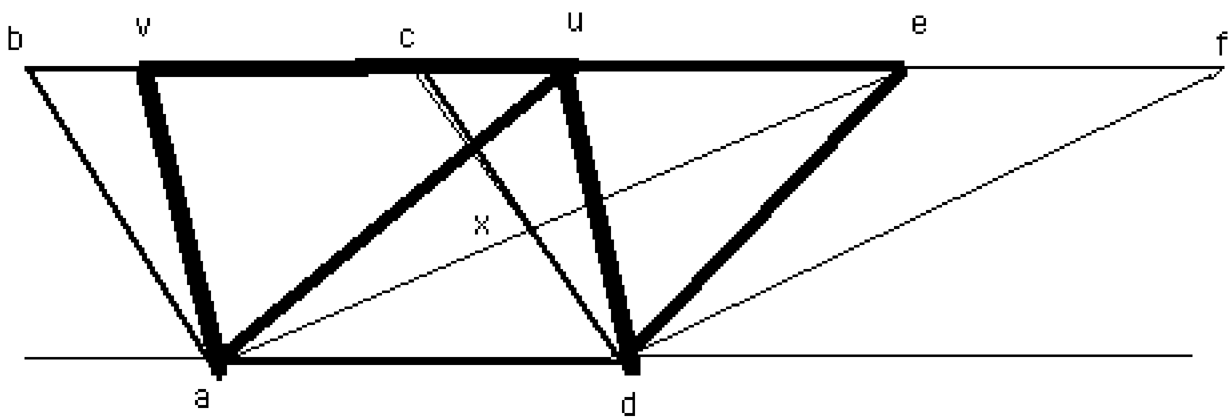
Remark: In fact the two parallelograms in the first case also have congruent dissections: We can even reduce that case to the easier case.



To show [abcd] and [aefd] have congruent dissections, it suffices by transitivity to find a parallelogram which has congruent dissections with both of these. Look at the next picture.



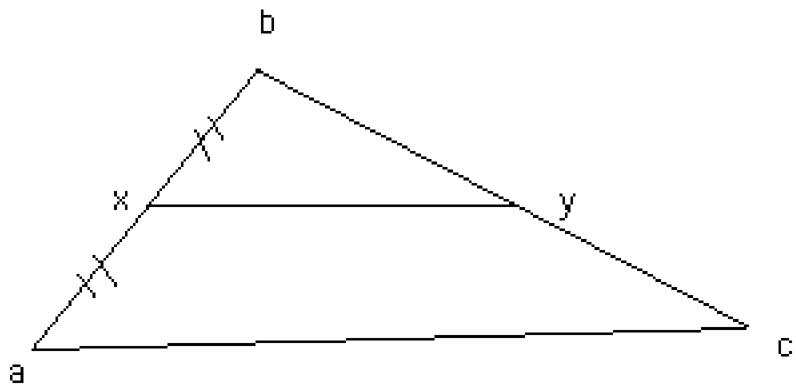
Parallelogram [aued] has a congruent dissection with [aefd] by the easy case since there is no point of intersection. But we can do this again.



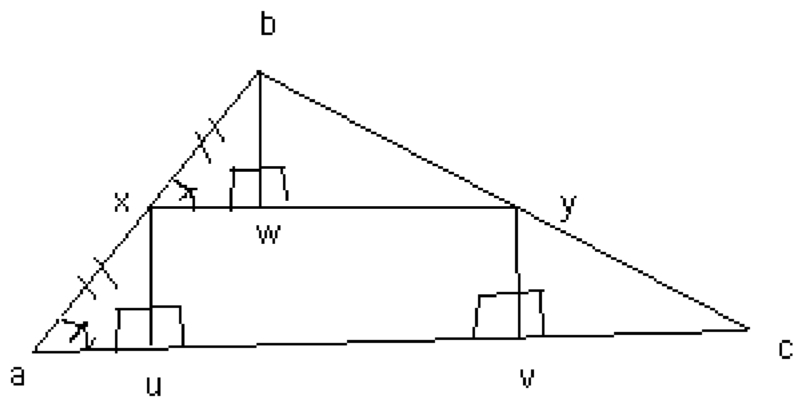
I.e. here [avud] has a congruent dissection with [aued] hence with [aefd]. Now we are back in the easy case, so [avud] has a congruent dissection with [abcd]. By transitivity, [abcd] has a congruent dissection with [aefd].
Hence we have proved (assuming transitivity):
Strong Prop 35: Parallelograms which are on the same base and "in the same parallels" have congruent dissections.

Next we use Hartshorne's argument to strengthen Euclid's Prop. 37.
Strong Prop. 37. Triangles which are on the same base and "in the same parallels" [i.e. whose vertices opposite the common base lie on a line parallel to the common base], have congruent dissections.

Lemma: A line through the midpoint of one side of a triangle, and parallel to the base, meets the third side at its midpoint.
proof:

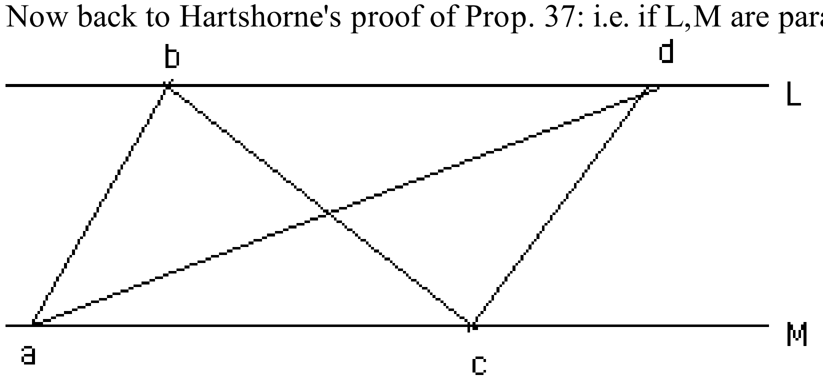


Assume x is the midpoint of ab, and xy is parallel to ac. Claim y is the midpoint of bc. Drop perpendiculars to form right triangles as follows:

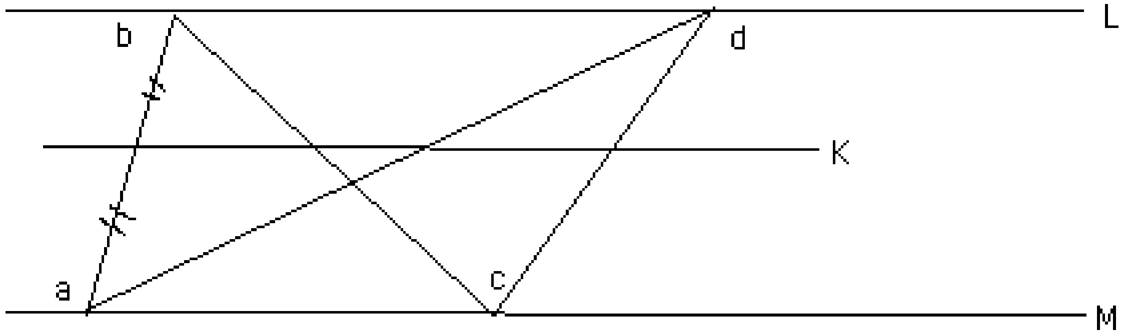


Triangles <axu> and <xbw> are congruent by hypotenuse angle since angles <xau and <bxw are equal by Z principle. Hence sides bw and xu are equal. By EFP all 4 angles of quadrilateral uxyv are 90, so sides xu and yv both equal bw. Triangles <bwy> and <yvc> are congruent by SAA, since angles wyb and vcy are equal by Z principle. Thus segments (by) and (yc) are equal, so y is the midpoint of bc. QED.

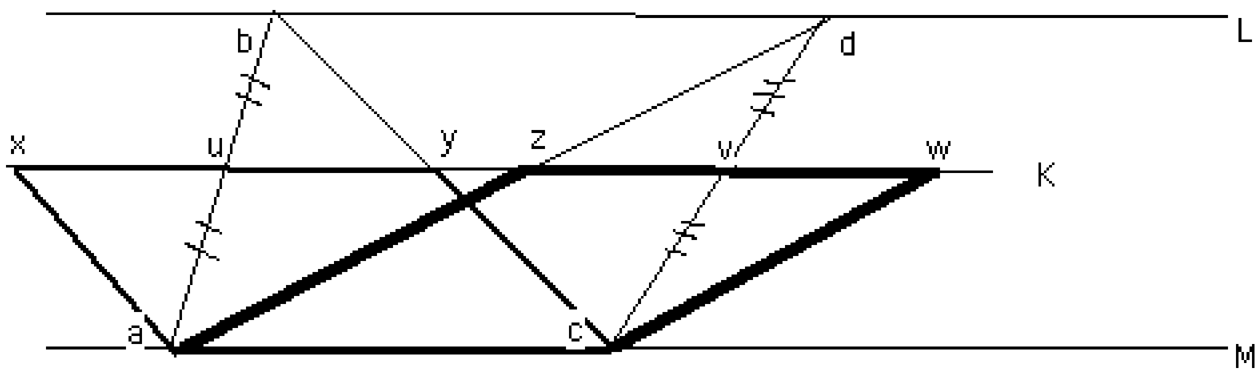
Now back to Hartshorne's proof of Prop. 37: i.e. if L,M are parallel,



then triangles <abc>, <adc>, have congruent dissections. To see this draw a line K parallel to L,M through the midpoint of side (ab).



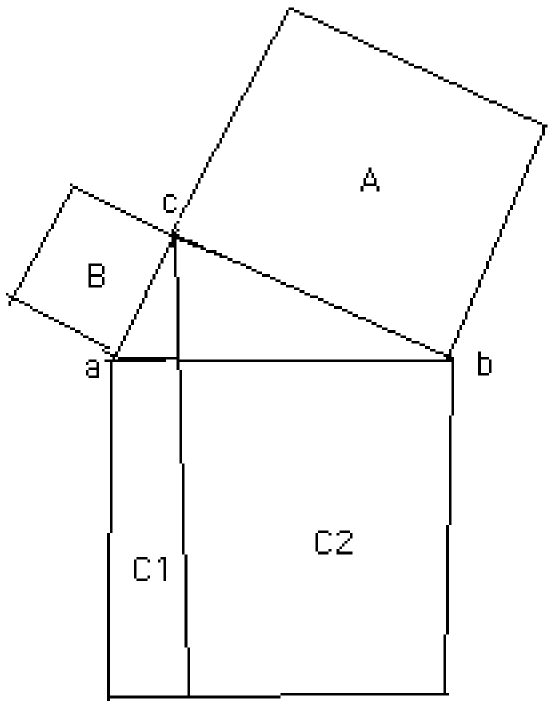
Then applying the lemma to triangles <abc>, <abd>, and <adc>, it follows that K bisects all the segments (ab, (ad), (bc), and (dc). Now construct parallelograms on the base (ac), with tops in K, as follows:



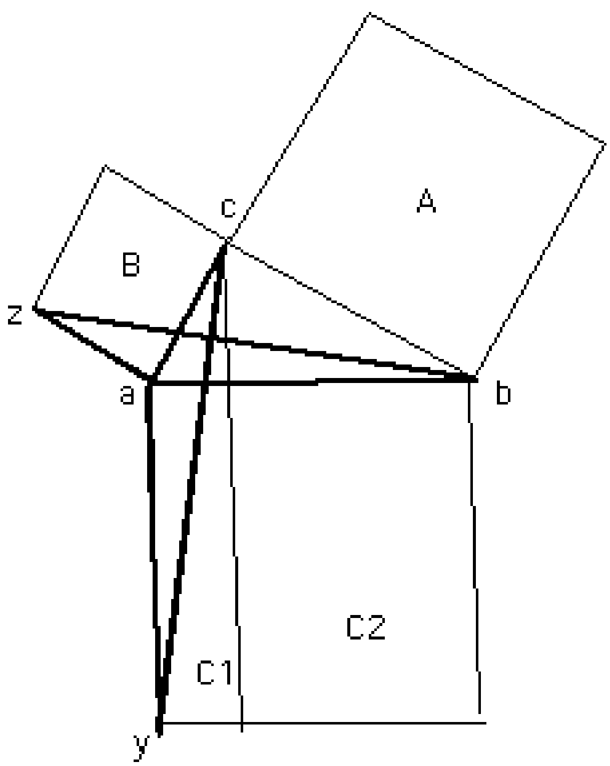
Triangles <aux> and <buy> are congruent, as are <dvz> and <cvw>. Hence triangle <abc> has a congruent dissection with parallelogram [axyc], and <adc> has a congruent dissection with parallelogram [azwc]. Those two parallelograms also have congruent dissections by Prop. 35, hence so do triangles <abc>, and <adc>. **QED Prop 37.**

Next we can give Euclid's proof of Pythagoras, and in a stronger form than he stated it, assuming as always, transitivity of the relation of having congruent dissections.

Prop. 47 (Pythagoras). If <abc> is a right triangle with hypotenuse (ab), then the sum of the squares A+B on the legs (ac) and (bc), has a congruent dissection with the square C on the hypotenuse (ab).
proof: Draw the squares on the various sides, and then drop a perpendicular from the vertex c (opposite the hypotenuse), onto the bottom edge of the square C on the hypotenuse, separating that square into two rectangles, C1 and C2. Then we will show that rectangle C1 has a congruent dissection with the square B on the side opposite vertex b, and rectangle C2 has a congruent dissection with the square A opposite the vertex a. Since sums of figures having congruent dissections obviously also have them, this will complete the proof.



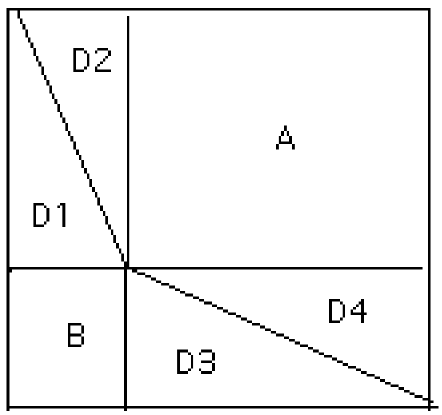
We want to show first rectangle C1 has congruent dissection with square B. To do this draw in two congruent triangles, <azb>, and <acy>.

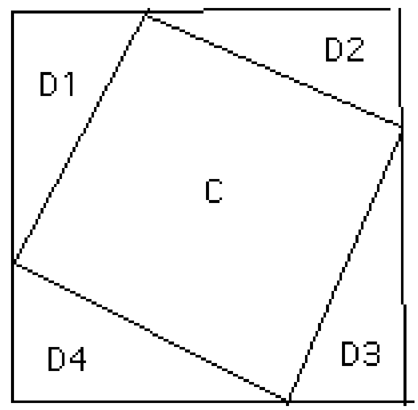


The triangles are congruent by SAS, since angles <zab and <cay both equal angle <cab plus a right angle, and the sides (ac) and (az) are sides of the same square, as are sides (ay) and (ab). Now triangle <acy> has a congruent dissection with the triangle formed by half of rectangle C1. by

Prop. 37, as does the triangle <azb> with half of square B. Since doubles of figures with congruent dissections also have congruent dissections, C1 has a congruent dissection with B. Similarly rectangle C2 has congruent dissection with square A. Adding, square C = C1+C2 has congruent dissection with the sum of squares A+B. **QED.**

The pictures below show A+B and C have equal content, since adding the same 4 congruent triangles to both figures gives the same result.





The next picture from your hw shows more easily than Euclid's argument that A+B and C also have congruent dissections, since the dark lines show how to cut 5 pieces from A and B, which can be reassembled to form C.