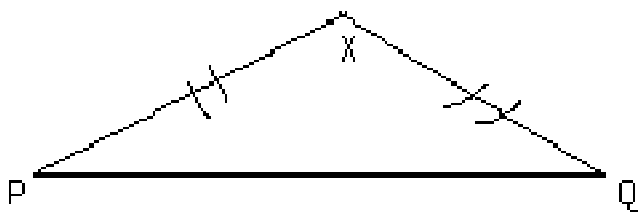
5200/7200 Fall 2007 Concurrence theorems for triangles

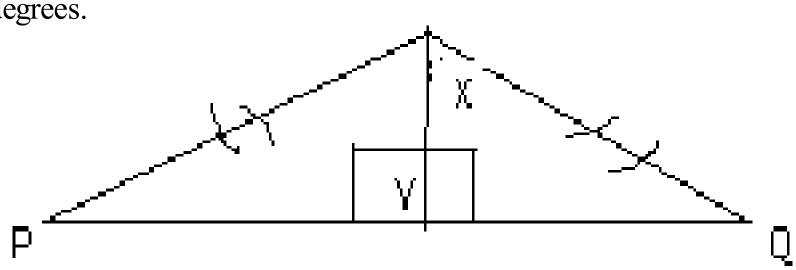
There are two basic concurrence theorems for triangles that hold in neutral geometry, that of medians and of angle bisectors, but it seems hard to prove the concurrence of medians. In particular I do not have an elementary proof. Thus we will only prove it for euclidean geometry, along with the two concurrence theorems peculiar to euclidean geometry, that of altitudes and perpendicular bisectors of sides.

Theorem A: In a Euclidean triangle, the perpendicular bisectors of the sides all meet in a common point, (which may not be inside the triangle).
proof:

Lemma 1: In a neutral geometry, a point X is on the perpendicular bisector of a segment PQ if and only if the distances d(X,P) and d(X,Q) are equal.
proof: If X is equidistant from P and Q, then triangle PXQ has two equal sides, PX and QX, hence is isosceles.

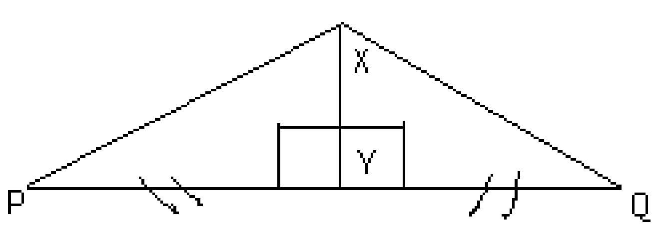


If the bisector of angle X meets the base PQ at Y, then triangles XYP and XYQ are congruent by SAS, so angles XYP and XYQ are equal and add to a straight angle, so each is 90 degrees.



I.e. XY is perpendicular to PQ, and also sides YP and YQ are congruent, so XY is the perpendicular bisector of PQ. Hence the point X which is equidistant from P,Q does lie on the perpendicular bisector of PQ.

Conversely, if X lies on the perpendicular bisector of segment PQ with midpoint Y, the triangles XYP and XYQ are congruent by SAS, hence also sides XP and XQ are congruent, so indeed X is equidistant from P and Q. (See picture on next page.)

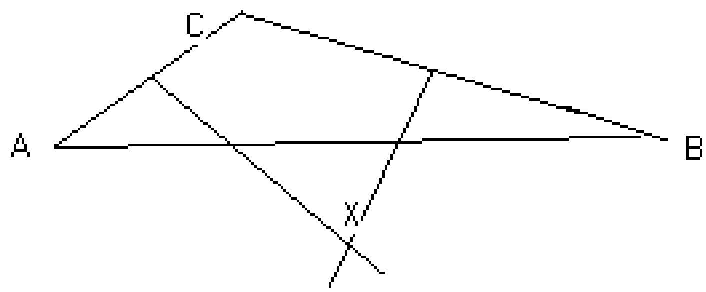


Thus d(X,P) = d(X,Q).

QED.

Lemma 2: In Euclidean geometry, if lines L,K are parallel, and if M is perpendicular to L and N is perpendicular to K, then M,N are also parallel.
proof: Since M is perpendicular to L at some point X, and since L is the only line through X which is parallel to K, M also meets K, and by the Z - principle, M is perpendicular to K. Hence M,N have a common perpendicular, and thus M,N satisfy the Z - principle hence are parallel.
QED.

Draw perpendicular bisectors of two sides, AC and BC, of triangle ABC.

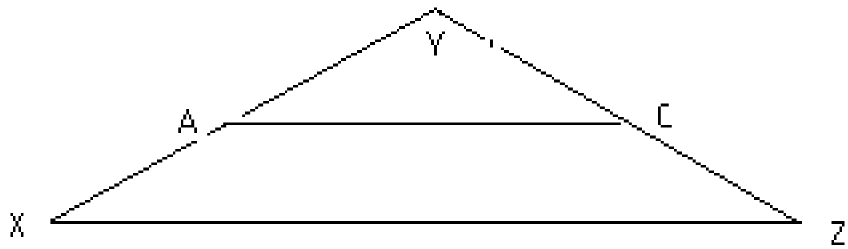


By lemma 2, since AC and BC are not parallel, the perpendicular bisectors cannot be parallel, hence meet at some point X, which is equidistant from A and C by lemma 1, and also from B and C. I.e. d(A,X) = d(C,X) = d(B,X), so X is also equidistant from A and B, hence X lies on the perpendicular bisector of side AB as well. **QED.**

Remark: Since lemma 1 holds in a neutral geometry, the proof shows that if two of the perpendicular bisectors of a neutral happen to meet, then all three meet at one point. So in neutral geometry, perpendicular bisectors of the sides of a triangle either meet in one point, or else they are all parallel! This latter case can happen in some hyperbolic triangles.

Theorem B: In Euclidean geometry, the altitudes of a triangle all meet in one point, (which again may not be inside the triangle).
proof: Given triangle ABC, construct its "double", a triangle XYZ whose sides are each twice as long as those of ABC. This is possible since we know from the two circles theorem that a triangle exists with sides of lengths x,y,z, if and only if no one of those numbers is greater than the sum of the other two. If that is true for the numbers a,b,c, which are the side lengths of triangle ABC, then it is also true for the numbers x=2a, y = 2b, z = 2c. Hence the double of a triangle does exist.

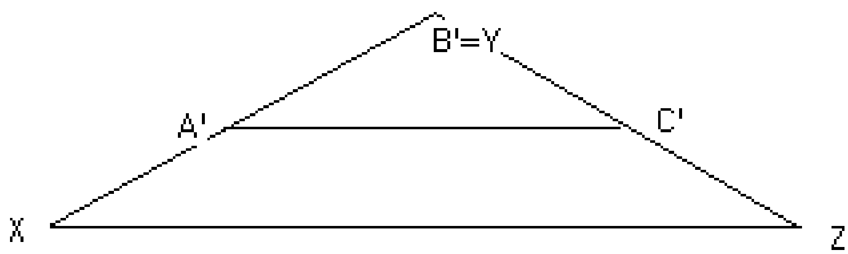
Lemma 3: In a triangle XYZ, let A,C be points on sides XY and YZ respectively. Then the ratios |AY|/|XY| and |CY|/|ZY| are equal if and only if the segment AC is parallel to the base XZ.



proof:

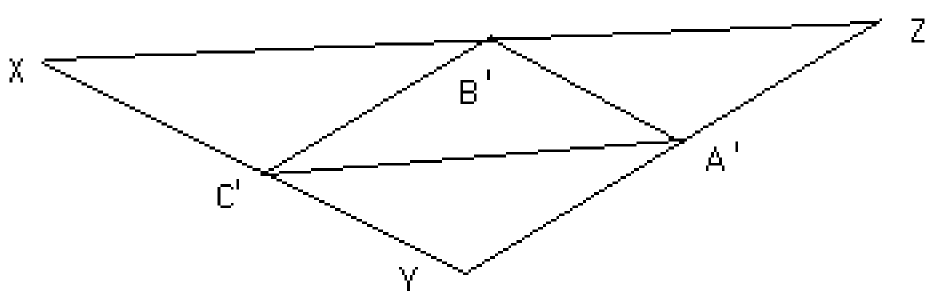
First assume AC is parallel to XZ. Then by the Z principle, angles YAC and YXZ are equal, and also angles YCA and YZX are equal. Thus triangles XYZ and AYC are similar and hence by the basic theorem of similarity theory, the ratios of corresponding sides are equal, so |AY|/|XY| = |CY|/|ZY|. Conversely, if AC is not parallel to XZ, then the line through A which is parallel to XZ cuts YZ at a different point C' from C. Then the ratios |AY|/|XY| and |C'Y|/|ZY| are equal, so since |CY| is different from |C'Y|, the ratios |AY|/|XY| and |CY|/|ZY| are not equal. **QED.**

Corollary: (SAS similarity) If two triangles ABC and XYZ are given, and if angle B = angle Y, and if the ratios |AB|/|XY| = |BC|/|YZ| are equal, then triangle ABC is similar to triangle XYZ.
proof: Assume |AB| < |XY|, hence also |BC| < |YZ|. By a rigid motion put point B at point Y, and align side AB with side XY, and side BC with side YZ, i.e. put triangle ABC inside of triangle XYZ.



Then by the previous proof, since the ratios |A'Y|/|XY| and |C'Y|/|ZY| are equal, A'C' is parallel to XZ, so angle YA'C' = angle YXZ, and angle YC'A' = angle YZX. Thus triangle A'B'C' is similar to triangle XYZ, and so is the congruent triangle ABC. **QED.**

Now given triangle ABC look at its double triangle XYZ, and join the midpoints A',B',C' of the three sides of XYZ.

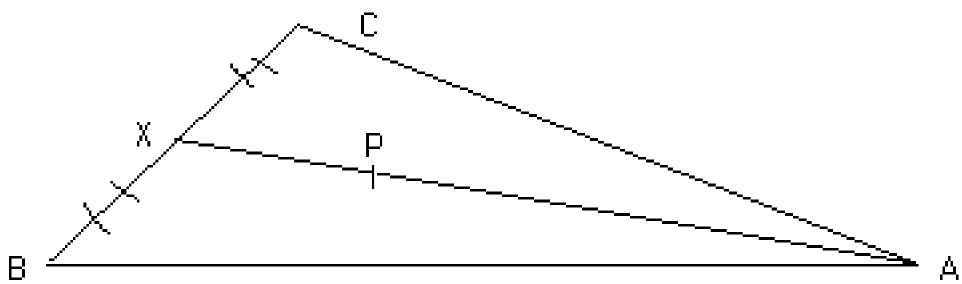


Claim: triangle A'B'C' is congruent to triangle ABC.
proof: By lemma 3, since |YA'|/|YX| = |YC'|/|YZ| = 1/2, triangle A'YC' is similar to triangle XYZ,

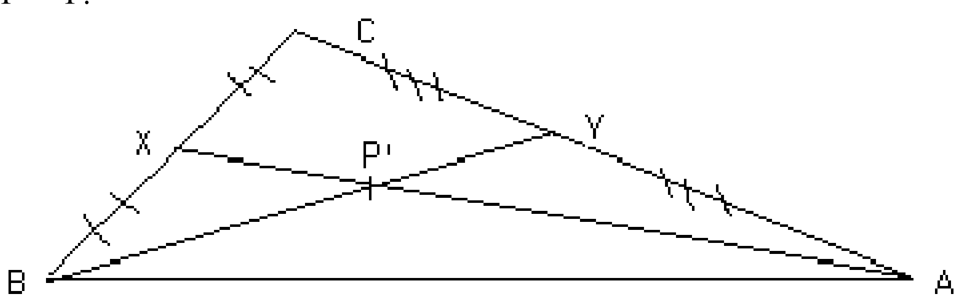
and side A’C’ is also 1/2 as long as side XZ. The same argument for each side shows the sides of triangle A’B’C’ are 1/2 as long as those of triangle XYZ, hence are equal to the sides of triangle ABC. Thus triangles ABC and A’B’C’ are congruent by SSS. QED.

Thus to show the altitudes of triangle ABC meet in one point, it suffices to show it for triangle A’B’C’. But the altitudes of triangle A’B’C’ are equal to the perpendicular bisectors of the sides of triangle XYZ. I.e. the altitude of A’B’C’ at B’ passes through the midpoint B’ of side XZ and is perpendicular to A’C’, hence is also perpendicular to the parallel line XZ, by the Z principle. Use the same argument for the other altitudes. **QED.**

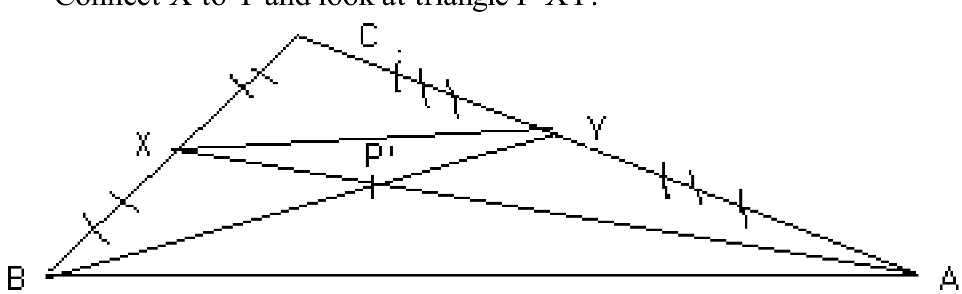
Theorem C: In a Euclidean geometry (and in a neutral geometry), the medians of a triangle ABC meet at one point, which is inside the triangle.
proof: (Euclidean case only) We will show that the medians meet at a point 2/3 of the way up each median from the vertex. So draw the median from A and let the point 2/3 of the way up that median from A be called P. We will show that the other medians also pass through P. For example we will show the median from B passes through P.



Draw the median from B and label the point where it meets AX as P’. We want to show P’ = P.



Connect X to Y and look at triangle P’XY.

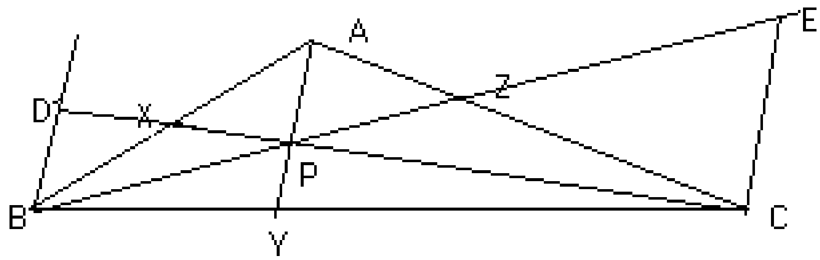


By lemma 3, triangle CXY is similar to triangle CBA, so XY is parallel to and 1/2 of side

BA. By Z-principle, triangle P'XY is similar to triangle P'AB, so XP' is half as long as P'A. Thus P' is 2/3 of the way from A, so P' = P. Similarly the median from C also passes through P. **QED.**

Remark: The point where the medians meet in a neutral geometry is not always 2/3 of the way up from the vertices.

"Ceva's" theorem: In a Euclidean triangle ABC, if points X,Y,Z lie on the sides opposite vertices C,A,B, then segments CX, AY, BZ are concurrent if and only if the product of ratios |AX|/|XB| |BY|/|YC| |CZ|/|ZA| = 1.
proof: Assume concurrence at P, and draw lines parallel to AY, from B and C, meeting CX (extended) at D, and meeting BZ (extended) at E.



Then triangles BXD and AXP are similar, as are triangles CZE and AZP. Hence |AX|/|BX| = |AP|/|BD|, and |CZ|/|AZ| = |CE|/|AP|. Also triangles BYP and BPE are similar as are triangles BCD and YCP. So |BY|/|BC| = |YP|/|CE|, and |BC|/|YC| = |BD|/|YP|. Multiplying these equations, product of LHS =
|AX|/|XB| |BY|/|YC| |CZ|/|ZA| = product of RHS = 1. Converse, exercise. **QED.**