2017 GRADUATE PRELIMINARY EXAM

All problems are weighted equally. Throughout \( \mathbb{R} \) denotes the real numbers and \( \mathbb{C} \) the complex numbers.

1. Negate the following statements in a "non-cheap" way – especially, avoid using the word "not."
   (a) For all real numbers \( x \), there is a real number \( y \) such that \( |x - y| \geq 2017 \).
   (b) The function \( f : \mathbb{R} \to \mathbb{R} \) is continuous.

2. Let \( V = \{(x, y, z) \in \mathbb{R}^3 \mid 3x + 4y + 5z = 0 \} \).
   (a) Show that \( V \) is a linear subspace of \( \mathbb{R}^3 \).
   (b) Prove or disprove: there is a linear transformation \( S : \mathbb{R}^3 \to \mathbb{R}^3 \) with kernel equal to \( V \).
   (c) Prove or disprove: there is a linear transformation \( T : \mathbb{R}^3 \to \mathbb{R}^3 \) with image equal to \( V \).
   (d) Prove or disprove: there is a linear transformation \( U : \mathbb{R}^3 \to \mathbb{R}^3 \) with kernel and image equal to \( V \).

3. Show (we suggest by induction) that for all non-negative integers \( n \), we have
   \[
   \int_0^\infty x^n e^{-x} dx = n!
   \]

4. For a positive integer \( n \), let \( I_n \) denote the \( n \times n \) identity matrix.
   (a) Let \( A \) be a \( 2 \times 2 \) real matrix with eigenvalues \( \lambda_1 = 1 \) and \( \lambda_2 = -1 \). Show: \( A^2 = I_2 \).
   (b) Find a \( 3 \times 3 \) real matrix whose only eigenvalues (in \( \mathbb{C} \)) are \( 1 \) and \( -1 \) such that \( A^2 \neq I_3 \).
   (c) Let \( A \) be an \( n \times n \) real symmetric matrix whose only eigenvalues (in \( \mathbb{C} \)) are \( 1 \) and \( -1 \). Show that \( A^2 = I_n \).

5. Let \( \{f_n : [0, 1] \to \mathbb{R}\}_{n=1}^\infty \) be a sequence of continuous functions that converges uniformly to 0. Show that the sequence \( \int_0^1 f_n(x) dx \) converges to 0.

6. Use the \( \epsilon, \delta \) definition of the limit to show: \( \lim_{x \to 1} \frac{x^2 + 1}{x} = 2 \).

7. Let \( z = f(x, y) \) be a smooth surface. Show that the gradient is perpendicular to the level curves.
   (Suggestion: let \( \gamma(t) \) be a curve contained in a level set of \( f \), and consider the derivative of \( f \circ \gamma \).

8. Let \( X \) and \( Y \) be sets and let \( f : X \to Y \) and \( g : Y \to X \) be functions. We suppose throughout that \( g(f(x)) = x \) for all \( x \in X \).
   (a) Show that \( f \) is injective.
   (b) Show that \( g \) is surjective.
   (c) Give an example in which neither \( f \) nor \( g \) is bijective.