Real Analysis Qualifying Examination
Fall 2014

The six problems on this exam have equal weighting.

1. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of continuous functions on $\mathbb{R}$ for which the series $\sum_{n=1}^{\infty} f_n$ converges uniformly. Prove that the sum function $f := \sum_{n=1}^{\infty} f_n$ is also continuous.

2. Let I be an index set and $a : I \to (0, \infty)$.

   (a) Show that if
   $$\sum_{i\in I} a(i) := \sup_{J\subset I, J \text{ finite}} \sum_{i\in J} a(i) < \infty,$$
   then I is countable.

   (b) Suppose $I = \mathbb{Q}$, and that $\sum_{q\in\mathbb{Q}} a(q) < \infty$. Show that the function f, defined for all $x \in \mathbb{R}$ by
   $$f(x) := \sum_{q\in\mathbb{Q}, q\le x} a(q),$$
   is continuous at x if and only if $x \notin \mathbb{Q}$.

3. Let f be a Lebesgue integrable function on $\mathbb{R}$. Prove that for every $\varepsilon > 0$ there exists $\delta > 0$ such that
   $$\int_E |f(x)|\, dx < \varepsilon$$
   whenever $m(E) < \delta$.

4. Let $g \in L^{\infty}([0, 1])$. Prove that if
   $$\int_{[0,1]} f(x)g(x)\, dx = 0$$
   for all continuous functions f on $[0, 1]$, then $g(x) = 0$ almost everywhere.

5. (a) Show that if f is continuous with compact support on $\mathbb{R}^n$, then
   $$\lim_{t\to 0} \int_{\mathbb{R}^n} |f(x + t) - f(x)|\, dx = 0.$$

   (b) Extend the result above to functions in $L^1(\mathbb{R}^n)$ and use this extension to prove that if $f \in L^1(\mathbb{R}^n)$ and $g \in L^{\infty}(\mathbb{R}^n)$, then
   $$f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy$$
   is both bounded and uniformly continuous.

6. Let $1 \le p, q \le \infty$ be conjugate exponents, i.e. $\frac{1}{p} + \frac{1}{q} = 1$. Prove that if $f \in L^p(\mathbb{R}^n)$, then
   $$\|f\|_p = \sup_{\|g\|_q=1} \left| \int f(x)g(x)\, dx \right|.$$