**Deformations of isolated even double points of corank one**

R. Smith and R. Varley
(University of Georgia)

We give a local deformation theoretic proof of Farkas' conjecture that a principally polarized complex abelian variety of dimension 4 whose theta divisor has an isolated double point of rank 3 at a point of order two is a Jacobian. The argument yields an explicit local normal form for the theta function near such a point. The proof depends only on the facts that the theta function is even, a general theta divisor is smooth, and a general singular theta divisor has only ordinary singularities.

## Introduction

Hershel Farkas conjectured [F] in 2004 the following statement designed to complete the geometric Schottky problem in genus 4: if the theta divisor on a 4 dimensional complex principally polarized abelian variety (ppav) $(A, \Theta)$ has an isolated double point of rank 3 at a point of order two for the group law, then $(A, \Theta)$ is a Jacobian of a smooth curve of genus 4 (which then has a vanishing even theta null). This was proved by Grushevsky and Salvati Manni in 2006 in [G-SM1] and completes the Andreotti - Mayer (and classical) program of characterizing Jacobians of genus 4 curves among all 4 dimensional ppav's by the singular points on $\Theta$.

**Characterization of genus 4 Jacobians**:

Let $A$ be a 4 dimensional complex ppav and $\Theta$ a symmetric theta divisor on $A$, and $sing(\Theta)$ its variety of singular points.

i) $A$ is a product of lower dimensional Jacobians, iff $sing(\Theta)$ has dimension 2.

ii) $A$ is a hyperelliptic Jacobian, iff $sing(\Theta)$ has dimension one.

iii) $A$ is a non hyperelliptic Jacobian with no vanishing even theta null, iff $\Theta$ has exactly 2 "conjugate" singularities, (inverses for the group law).

iv) $A$ is a non hyperelliptic Jacobian with a vanishing even theta null, iff $\Theta$ has an isolated rank 3 double point at a point of order two.

The "only if" statements are classical, and the "if" statements are due in parts i), ii), iii) to Beauville, and in iv) to Grushevsky and Salvati Manni. See [B, Thm. p.149, (6.6) p.181, (7.4) p.184, (7.5) p.191] and [G-SM1].

In this paper we prove a local structure theorem for theta functions which implies the following statement in genus 4:

**Proposition**: Locally near an isolated rank 3 double point (x;s) = (0;0) of the fiber $\vartheta(x; 0) = 0$ over $s = 0$ in $\mathbb{C}^4 \times \mathcal{H}_4$, the ideal of the universal theta function of 4 variables is generated (after equivariant change of variables) by a polynomial of form $x_1^2 + x_2^2 + x_3^2 + x_4^4 + b(s)x_4^2 + c(s)$, where $b, c$ are analytic functions on Siegel space $\mathcal{H}_4$ near $s = 0$ such that $b(0) = c(0) = 0$. Moreover no component of the divisor $\{b^2(s) - 4c(s) = 0\}$ is contained in the divisor $\{c(s) = 0\}$.

This implies Farkas' conjecture as follows:

**Corollary**: A four dimensional ppav $(A, \Theta)$ whose theta divisor has an isolated double point of rank 3 at a point of order two is a Jacobian of a smooth non hyperelliptic curve of genus 4 with a vanishing even theta null.

**Proof of Corollary**: By the proposition, in every neighborhood of $s = 0$, there are points $s$ in $\mathcal{H}_4$ with $b^2(s) - 4c(s) = 0$ and $c(s) \neq 0$, over which the theta divisor has two distinct non-zero ordinary double points (odp's) $\{x, -x\}$ as singularities. Since there is only one component of the discriminant locus $\mathcal{N}_0$ in $\mathcal{A}_4$ over which the general singular theta divisor has more than one singularity, namely $\mathcal{J}_4 = \{$Jacobians and products of Jacobians$\}$, $(A, \Theta)$ lies on $\mathcal{J}_4$. But $(A, \Theta)$ is neither a product of Jacobians nor a hyperelliptic Jacobian, since on those ppav's $\Theta$ has no isolated singularities. The double point of order two then represents the vanishing even theta null on the corresponding smooth non hyperelliptic curve. **QED**

## Motivation from deformation theory of singularities

Although the argument does not appeal to general theorems of deformation theory, it is suggested by them. One knows that every family of local hypersurface singularities which specializes to a given isolated singularity, is pulled back by a classifying map from one standard model family, the versal family for that singularity. A double point of "corank one" is one of the simplest singularities, defined by a function analytically equivalent to one of form $(x_n^2 + ... + x_1^2 + x_0^m)$, for $m > 2$, and the versal deformation of this singularity is equivalent to that of the monomial $x_0^m$, i.e. to a generic monic polynomial of degree $m$, [A-G-V, pp.187-188]. If the function is also even, and the singularity is at $x = 0$, the model is $(x_n^2 + ... + x_1^2 + x_0^{2k})$, for some $k \geq 2$, and the original family is equivalent by the argument below to one of this form: $(x_n^2 + ... + x_1^2 + x_0^{2k} + a_{k-1}(s)x_0^{2(k-1)} + ... + a_1(s)x_0^2 + a_0(s))$, where all $a_j(0) = 0$. A reference for hypersurface deformations in the analytic case is [K-S], and a reference for deformations of even hypersurfaces in the formal case is [R].

Thus for a theta function with an isolated double point of corank one, the model is that of a generic even polynomial $y^{2k} + a_{k-1}(s)y^{2(k-1)} + ... + a_1(s)y^2 + a_0(s)$, specializing to the monomial $y^{2k}$ for $s = 0$. The discriminant locus of this polynomial has two components: one is the locus $a_0 = 0$, where generically there is one odp at $y = 0$; the other is the pullback of the discriminant locus of the polynomial $t^k + a_{k-1}(s)t^{(k-1)} + ... + a_1(s)t + a_0(s)$, under the map $y \mapsto y^2 = t$, where generically there are 2 odp's which are negatives of each other.

In some sense this explains why $\mathcal{N}_0$ has two components and the structure of generic singularities of theta on those components. In particular, near a ppav with an isolated double point of corank one at 0 on theta, the family of all ppav's is locally the pullback of the general "even" deformation of the singularity $y^{2k}$ by a classifying map whose image meets a generic point of both components of the discriminant locus.

## Proof of the proposition

**Step one**: We construct a local classifying map, using Weierstrass preparation to produce a polynomial generator for the ideal of the theta divisor locally near the given singularity.

**Preparation Lemma**: Let $F(x; s)$ be analytic near $(0; 0)$ in $\mathbb{C}^{n+1} \times \mathbb{C}^r$ and "even in x", i.e. $F(-x; s) = F(x, s)$, and let $F(x; 0)$ have a "rank $n$ (or corank one) double point" at $x = 0$, (where if $n = 0$, this means $F(x_0; 0)$ has no terms of degree two or less in $x_0$).

Then there is an analytic coordinate system $(z; s) = (z(x; s); s)$ near $(0; 0)$, which is equivariant for the minus map in $x$, i.e. $z(-x, s) = -z(x; s)$, such that in these coordinates $F$ has the form:

$F(z; s) = F(x(z; s); s) = unit \cdot (z_n^2 + ... + z_1^2 + g(z_0; s))$, where the unit is analytic in $(z; s)$ and even in $z = (z_0, ..., z_n)$, and $g$ is analytic in $(z_0; s)$ and even in $z_0$. If moreover the point $x = 0$ is an isolated singularity of $F(x; 0)$, then the function $g$ may be taken to be a polynomial in $z_0$: $g(z_0; s) = z_0^{2k} + \sum_{0 \leq j \leq k-1} a_j(s)z_0^{2j}$, for some finite $k \geq 2$, with analytic coefficients $a(s)$, such that all $a_j(0) = 0$.

**Proof**: After a linear change of the coordinates $x$ in $\mathbb{C}^{n+1}$, which we continue to denote by $x$, we may arrange the homogeneous quadratic term of $F(x; 0)$ to be $x_n^2 +$ a rank $n - 1$ quadratic in $(x_0, ..., x_{n-1})$. Then by Weierstrass preparation [G-R, p.68], there is a unique expression $F(x; s) = unit \cdot (x_n^2 + b_1x_n + b_0)$ where $b_1$ and $b_0$ are analytic on a neighborhood of $(0; 0)$ in $\mathbb{C}^n \times \mathbb{C}^r$ and vanish at $(0; 0)$. By the uniqueness of this Weierstrass polynomial, and the evenness of $F$ in $x$, it follows that $b_1$ is odd in $x$, and both $b_0$ and the unit are even in $x$. Now complete the square in the variable $x_n$, by putting $y_n = x_n + b_1/2$, so that $F(x_0, ..., x_{n-1}, y_n; s) = unit \cdot (y_n^2 + c_0)$ where $y_n$ is odd in $x$, $c_0(x_0, ..., x_{n-1}; s) = b_0 - b_1^2/4$ is even in $x$, and $c_0(x; 0)$ has a rank $n - 1$ double point at $x = 0$.

Next repeat the argument for $c_0$ that was given for $F$. I.e. change $(x_0, ..., x_{n-1})$ linearly again until the homogeneous quadratic term of $c_0(x; 0)$ begins with $x_{n-1}^2$, use Weierstrass, complete the square replacing $x_{n-1}$ by $y_{n-1}$, to get $F(x_0, ..., x_{n-2}, y_{n-1}, y_n; s) = unit \cdot (y_n^2 + unit \cdot (y_{n-1}^2 + d_0(x_0, ..., x_{n-2}; s)))$ where both units and $d_0$ are even in $x$, and $y_{n-1}$ and $y_n$ are both odd in $x$. Then divide $y_n$ by an analytic, hence even, square root of the inner unit, replacing $y_n$ by $z_n$, which is thus still odd in $x$.

Then $F(x_0, ..., x_{n-2}, y_{n-1}, z_n; s) = unit \cdot (z_n^2 + y_{n-1}^2 + d_0(x_0, ..., x_{n-2}; s))$, where the outer unit is the product of the two previous units, and $d_0$ is even in $x$, and $d_0(x; 0)$ has a rank $n - 2$ double point at $x = 0$.

Continuing in this way, repeatedly changing the $x$'s linearly, applying Weierstrass, replacing the $x$'s with $y$'s by completing the square, and then by $z$'s after dividing by square roots of units, we eventually come to an expression $F(x_0, y_1, z_2, ..., z_n; s) = unit \cdot (z_n^2 + ... + z_2^2 + y_1^2 + g(x_0; s))$, where $y_1$ and all the $z_j$ are odd in $x$, and $g$ is even in $x_0$. If $g(x_0; 0)$ is identically zero, we let $z_1 = y_1, z_0 = x_0$, and stop here, noting that the singular locus of $F(x; 0)$ is the smooth curve $(x_0, 0, ..., 0)$.

If $g(x_0; 0)$ is not identically zero, but vanishes at $x_0 = 0$ to finite order $2k \geq 4$, then we may apply Weierstrass again to write $g(x_0; s) = unit \cdot (x_0^{2k} + \sum_{0 \leq j \leq k-1} a_j(s) x_0^{2j})$, where the evenness of the Weierstrass polynomial in $x_0$ and of the unit, follows from the evenness of $g$ in $x_0$, and the uniqueness of the Weierstrass polynomial and of the unit. Dividing $y_1$ by a square root of the unit replaces it by $z_1$ so that, again with a new unit out front, $F(x_0, z_1, ..., z_n; s) = unit \cdot (z_n^2 + ... + z_1^2 + x_0^{2k} + \sum_{0 \leq j < k} a_j(s) x_0^{2j})$, for some finite $k \geq 2$. Renaming $x_0 = z_0$, we have our result. **QED**

**Step two**: Next we deduce a general result for theta functions with a corank one double point at a point of order two. $\mathcal{H}_g$ denotes Siegel space of genus $g$, and $\mathcal{A}_g = \mathcal{H}_g / Sp(g, \mathbb{Z})$ the moduli variety of $g$ dimensional ppav's. We assume as known that the discriminant locus $\mathcal{N}_0$ parametrizing ppav's in $\mathcal{A}_g$ with singular theta divisor, has exactly two irreducible components for $g \geq 4$, called $\mathcal{N}_0'$ and $\theta_{\text{null}}$, and that a general theta divisor over $\mathcal{N}_0$ has only odp's as singularities, two of them over $\mathcal{N}_0'$ and one of them over $\theta_{\text{null}}$ [B, D1, G-SM2, S-V1,2].

**Theorem**: (i) Locally near an isolated corank one double point at $(x; s) = (0; 0)$ of the fiber $\vartheta(x; 0) = 0$ over $s = 0$ in $\mathbb{C}^g \times \mathcal{H}_g$, the ideal of the universal theta function of $g \geq 4$ variables is generated (after an equivariant change of variables) by a polynomial of form $x_1^2 + ... + x_{g-1}^2 + x_g^{2k} + \sum_{0 \leq j < k} a_j(s) x_g^{2j}$, where $k \geq 2$, and the $a_j(s)$ are analytic functions on $\mathcal{H}_g$ near $s = 0$ with $a_j(0) = 0$.
(ii) If $\Delta(a)$ is the discriminant function of the polynomial $t^k + \sum_{0 \leq j < k} a_j t^j$, then no component of the divisor $D_1^* = \{\Delta(a(s)) = 0\}$ is contained in the divisor $D_0^* = \{a_0(s) = 0\}$.
(iii) In particular in every neighborhood of $s = 0$, there are points $s$ in $\mathcal{H}_g$ with $\Delta(a(s)) = 0$ and $a_0(s) \neq 0$, over which the theta divisor has at least two distinct non zero points $\{x, -x\}$ as singularities. Hence $s = 0$ lies on both components of $\mathcal{N}_0$.

**Proof**: Part (i) follows from the preparation lemma and the evenness of $\vartheta$ at points of order two and even multiplicity on $\Theta$. Hence there is a neighborhood of $(0; 0)$ in $\mathbb{C}^g \times \mathcal{H}_g$ such that the critical locus of the restriction of $\vartheta$ to this neighborhood is locally isomorphic to a neighborhood of $(x; s) = (0; 0)$ in the critical locus of an analytic family of polynomials of form $(x_1^2 + ... + x_{g-1}^2 + x_g^{2k} + \sum_{0 \leq j < k} a_j(s) x_g^{2j})$, for some finite $k \geq 2$. The Jacobian criterion shows that the critical locus of this family is isomorphic to that of the family of even monic polynomials $x_g^{2k} + \sum_{0 \leq j < k} a_j(s) x_g^{2j}$ of the variable $x_g$. In particular $(x_1, ..., x_{g-1}, x_g)$ is a singular point for $(x_1^2 + ... + x_{g-1}^2 + x_g^{2k} + \sum a_j(s) x_g^{2j})$ if and only if $x_1 = ... = x_{g-1} = 0$, and $x_g$ is a singular point of the polynomial $f(x_g) = x_g^{2k} + \sum a_j(s) x_g^{2j}$.

Thus the critical locus we want to analyze near our isolated corank one double point, is locally the pull-back by the coefficient functions $a_j(s)$, of the critical locus of the generic even polynomial $y^{2k} + \sum_{0 \leq j < k} a_j y^{2j}$, near the singular point $y = 0$, of $y^{2k}$, i.e. near the point $(y; a_0, ..., a_{k-1}) = (y; a) = (0; 0)$ in $\mathbb{C} \times \mathbb{C}^k$. Thus we recall next the structure of the critical and discriminant loci of this model, the monic even polynomials of one variable.

Let $V =$ the space of monic even polynomials $y^{2k} + \sum_{0 \leq j < k} a_j y^{2j}$ of degree $2k$ in the variable $y$, parametrized by $(a_0, ..., a_{k-1})$ in $\mathbb{C}^k \cong V$. The discriminant locus is the subset $D$ in $V$ of those polynomials having a repeated root, and the critical locus is the subset in $\mathbb{C} \times \mathbb{C}^k$ of pairs $(y; a)$ where $y$ is a repeated root of the polynomial with coefficient vector $a$. We deduce that $D$ has exactly two irreducible components as follows. The singular points, or repeated roots, of a given even polynomial $f(y) = h(y^2)$ are the common zeroes of $h(y^2) = 0 = \frac{d}{dy} h(y^2) = 2y \cdot h'(y^2)$.

Thus if $h(t)$ has degree $k$ in $t$, the singular points of the even polynomial $h(y^2)$ of degree $2k$ in $y$, consist of the point $y = 0$ for those $h(t)$ with zero constant term, and the two square roots $y, -y$, of singular points $t$ of the polynomial $h(t)$. The discriminant locus of polynomials $h(t)$ of degree $k$ is known classically to be irreducible [cf. S-V3] and the space of polynomials with zero constant term is parametrized by the irreducible space $\mathbb{C}^{k-1}$. Thus the discriminant locus of even polynomials $h(y^2)$ of degree $2k$ has two irreducible components, $D_0$ = those $h(y^2)$ with zero constant term, and $D_1$ corresponding to those $h(y^2)$ where $h(t)$ has a repeated root.

As remarked above, the critical locus of this model family consists of pairs $(y; a)$ in $\mathbb{C} \times \mathbb{C}^k$ such that $y$ is a singular point of the polynomial with coefficient vector $a$. Over a generic point of $D_0$, we have $h(y^2)$ where $h(t)$ is a generic polynomial with zero constant term hence no multiple roots. Thus the only multiple root of $h(y^2)$ for such $h$ is $y = 0$. Hence the projection map $\mathbb{C} \times \mathbb{C}^k \to \mathbb{C}^k$ restricted to the critical locus has degree one over $D_0$. Over a generic point of $D_1$, we have $h(y^2)$ where $h(t)$ is a singular polynomial with only one repeated root $t$ which is non zero, so the repeated roots of $h(y^2)$ are then precisely the two square roots of $t$. Thus over $D_1$ the projection from the critical locus has degree two.

Of special interest to us is the neighborhood of the point $(y; a) = (0; 0)$ in the critical locus, i.e. of the singular point $y = 0$ for the monomial $y^{2k}$. Since this monomial has only one root, this is the only singular point, and since $k \geq 2$ by hypothesis, this monomial lies on both components $D_1$ and $D_0$ of the discriminant locus. We want to examine the singularities over the component $D_1$ near this point.

A polynomial in $D_1$ has form $f(y) = h(y^2)$ where $h(t)$ has a repeated root. Hence if $h(t)$ has a repeated root at a non zero number $t$, then both square roots of $t$ are singular points of $f(y) = h(y^2)$ so the critical locus has at least two points over this polynomial $f(y)$. Hence the only polynomials in $D_1$ having only one critical point over them, are those of form $h(y^2)$ where $h(t)$ has a singular point at $t = 0$, and nowhere else. This means both the constant term and linear term of $h(t)$ are zero, hence $h(t)$ is divisible by $t^2$, so $h(y^2)$ is divisible by $y^4$. In particular, if $f(y) = h(y^2)$ on $D_1$ has only one critical point, it occurs at $y = 0$, which is a point of multiplicity $\geq 4$.

Since the polynomial $y^{2k}$ has only one singularity, at $y = 0$, and the projection map from the critical locus to the parameter space V of even polynomials is proper, hence all polynomials near this one have all their singularities near $y = 0$. I.e. given any open disc $I$ around $y = 0$, there is a neighborhood of $y^{2k}$ such that all polynomials in this neighborhood have all their singularities in $I$.

Since $y^{2k}$ lies on both $D_0$ and $D_1$ in $V$, and a general theta divisor over $\mathcal{H}_g$ is smooth, the pullbacks $D_0^*$ and $D_1^*$ of $D_0, D_1$ by the map $s \mapsto a(s)$ are both non empty (possibly reducible), divisors through 0 in $\mathcal{H}_g$. Since theta has singularities over all points of $D_1^*$ and $D_0^*$, both $D_0^*$ and $D_1^*$ are contained in the discriminant locus $\mathcal{N}_0$ of $\mathcal{H}_g$. According to the definitions, $D_0^*$ is contained in $\theta_{null}$. To complete the proof of part (ii), we will show that no component of $D_1^*$ lies in $\theta_{null}$.

We see this as follows: since a component $Z$ of $D_1^*$ has the same dimension as $\mathcal{N}_0$, if $Z$ lies in $\theta_{null}$ it would contain a generic point of $\theta_{null}$, hence the singularity of theta over a generic point of $Z$ would be a single odp, since that is the generic singularity over $\theta_{null}$. But over $D_1^*$ and near $s = 0$, the critical locus of theta contains an isomorphic copy of the full critical locus of some polynomial $f(y) = y^{2k} + \sum_{0 \leq j < k} a_j y^{2j} = h(y^2)$ in $D_1$. If this were only one point, we remarked above then the corresponding critical point over $D_1$ would be at $y = 0$ and of multiplicity $\geq 4$ for $f$, since $f(y) = h(y^2)$ and $h(t)$ is a polynomial with singularity at $t = 0$. Hence the singularity of theta would be that of the polynomial $(x_1^2 + ... + x_{g-1}^2 + f(x_g))$ where $x_g^4$ divides $f(x_g)$. Then the singularity at $x = 0$ would be a corank one double point, and not an odp, which contradicts the known structure of singularities over a general point of $\theta_{null}$. Thus every component $Z$ of $D_1^*$ is contained in the other component $\mathcal{N}_0'$ of $\mathcal{N}_0$, where the generic singularity on theta is two odp's. This finishes the proof of (ii). Finally, since the point $s = 0$ lies on some such component $Z$ of $D_1^*$, the point $s = 0$ lies on both components of $\mathcal{N}_0$, and part (iii) follows. **QED**

**Remark**: Since the map from critical to discriminant locus in the model family of polynomials was proper, the non-zero singularities $\{x, -x\}$ produced in the previous argument over a general point of $D_1^*$ converge to $(x; s) = (0; 0)$, as $s \to 0$ in $\mathcal{H}_g$. Hence the point $s = 0$ actually lies on that component of the intersection of the two components of $\mathcal{N}_0$ called $R_g$ in [D1, pp.706-707]. Thus an isolated double point of corank one at a point of order two is a limit of the two ordinary double points on some nearby singular theta divisors.

**Step three**: Now we can deduce the proposition originally announced. Note that when $g = 4$, $\mathcal{N}_0' = \mathcal{J}_4 =$ {genus 4 Jacobians and products of lower genus Jacobians}.

**Proposition**: Locally near an isolated rank 3 double point at the origin $(x; s) = (0; 0)$ in $\mathbb{C}^4 \times \mathcal{H}_4$, the ideal of the universal theta function of 4 variables is generated by the polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^4 + bx_4^2 + c$, where $b(s), c(s)$ are analytic functions on Siegel space $\mathcal{H}_4$ near $s = 0$, such that $b(0) = c(0) = 0$. Moreover no component of the divisor $\{b^2(s) - 4c(s) = 0\}$ is contained in the divisor $\{c(s) = 0\}$.

**Proof**: By parts i) and ii) of the theorem for $g = 4$, it remains only to compute $k$ for the polynomial $(x_1^2 + x_2^2 + x_3^2 + x_4^{2k})$ which defines the genus 4 Jacobian theta divisor near the vanishing theta null. By the local algebra definition [A-G-V, pp.121,242], the Milnor number of this isolated singularity equals $2k - 1$. For a Jacobian with one vanishing even theta null, we can compute this number globally topologically to be 3 as follows [cf. S-V1]. The sum of the Milnor numbers of all the singularities of $\Theta$ (the global Milnor number) equals the difference between the Euler characteristic of $\Theta$ and the Euler characteristic of a generic smooth theta divisor. Hence the difference between the global Milnor numbers of the theta divisors of a generic Jacobian and a Jacobian with one vanishing even theta null equals the difference between their Euler characteristics. Comparing the Abel maps onto these two theta divisors shows this difference is one. Since a generic Jacobian theta divisor has just two odp's as singularities, the Milnor number of the theta divisor of a Jacobian with one vanishing even theta null is 3.

It follows that when $g = 4$ the polynomial model for the rank 3 double point has $k = 2$. Thus near the point $(x; s) = (0; 0)$, which is singular on the theta divisor of a genus 4 Jacobian with a vanishing even theta null, in suitable coordinates the theta function is associate in the local ring, to a polynomial $x_1^2 + x_2^2 + x_3^2 + x_4^4 + b(s)x_4^2 + c(s)$, with coefficients $b, c$, analytic on $\mathcal{H}_4$. Since $b^2 - 4c = 0$, and $c = 0$ define the discriminant loci $D_1$ and $D_0$ in the space of even polynomials of degree 4, they also define the pullbacks locally in $\mathcal{H}_4$, namely $D_1^*$ in $\mathcal{J}_4$, and $D_0^*$ in $\theta_{null}$. **QED**

## Comparisons and generalizations

The proof of Farkas' conjecture by Grushevsky and Salvati Manni in [G-SM1] is a global one, establishing a containment relation between the closure of the set of genus 4 Jacobians with vanishing even theta null and the set of 4 dimensional ppav's with a double point of rank $< 4$ at a point of order two, comparing the degree of these varieties in a projective embedding, and invoking Bezout's theorem to conclude equality as sets.

Since their argument treats all ranks less than 4, using [CM, Thm.3] it implies that that on an indecomposable 4 dimensional ppav $A$, the only possible singularities of theta are double points; and when theta is singular, an indecomposable $A$ fails to be a Jacobian if and only if theta has a double point of rank 4 at a point of order two, if and only if all double points of theta have rank 4 and occur at points of order two. Moreover, by [D2,V], there are at most ten of these isolated rank 4 vanishing theta nulls, and the maximum number occurs on a unique 4 dimensional ppav.

In higher dimensions, it is proved in [G-SM2] that if $g \geq 4$, the set of ppav's of dimension $g$ whose theta divisor has a point of order two with positive corank, equals Debarre's non-reduced component $R_g$ of the intersection of the two irreducible components of $\mathcal{N}_0$. Their argument uses the heat equation satisfied by the theta function to show that the equations defining the two loci are the same. In a forthcoming work we propose to strengthen our local arguments to remove the hypothesis of corank one used here, and to deduce these lower rank and higher dimensional results as well from general principles valid for all general even analytic hypersurfaces.

## References


[A-M] A. Andreotti and A. Mayer, On period relations for abelian integrals on algebraic curves, Ann. Scuola Norm. Sup. Pisa 21 (1967), 189-238.

[A-G-V] V. Arnol'd, S. Gusein - Zade, A. Varchenko, Singularities of Differentiable Maps, vol.I, Monographs in Mathematics, Birkhäuser 1985, Boston Basel Stuttgart.

[B] A. Beauville, Prym varieties and the Schottky problem, Invent. Math. 41 (1977), 149-196.

[CM] S. Casalaina-Martin, Cubic threefolds and abelian varieties of dimension five, II, Math. Z. 260 (2008), no. 1, 115–125.

[D] O. Debarre: (1) Le lieu des variétés abéliennes dont le diviseur thêta est singulier a deux composantes, Ann. Sc. École Norm. Sup. 25 (1992), 687-708; and (2) Annulation de thêtaconstantes sur les variétés abéliennes de dimension quatre, C.R. Acad. Sci. Paris Sér. I Math. 305 (1987), no. 20, 885-888.

[F] H. Farkas, Vanishing theta nulls and Jacobians, in The geometry of Riemann Surfaces and Abelian Varieties, III Iberoamerican Congress on Geometry in honor of professor Sevín Recillas Pishmish's 60th birthday, editors J. Muñoz Porras, S. Popescu, and R. Rodríguez, Contemporary Mathematics, vol. 397, Amer. Math. Soc., 2006.

[G-SM] S. Grushevsky and R. Salvati Manni, (1) Jacobians with a vanishing theta null in genus 4, Israel J. Math. 164 (2008), 303-315; and (2) Singularities of the theta divisor at points of order two, Int. Math. Res. Not. IMRN, 2007, no.15, Art. ID, rnm 045, 15 pages.

[G-R] R. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, Prentice-Hall, 1965.

[K-S] A. Kas and M. Schlessinger, On the versal deformation of a complex space with an isolated singularity, Math. Ann. 196 (1972), 23-29.

[R] D. Rim, Equivariant G-structure on versal deformations, Trans. Amer. Math. Soc. 257 (1980), no.1, 217-226.

[S-V] R. Smith and R. Varley, (1) On the geometry of $\mathcal{N}_0$, Rend. Sem. Mat. Univers. Politecn. Torino 42, 2(1984), 29-37; (2) Components of the locus of singular theta divisors of genus 5, Algebraic Geometry, Sitges 1983, Springer Lecture Notes 1124, 338-416; and (3) The tangent cone to the discriminant, Proceedings of the 1984 Vancouver Conference in Algebraic Geometry, Pub. for Can. Math. Soc. by A.M.S., Providence, 1986, 443-460.

[V] R. Varley, Weddle's surfaces, Humbert's curves, and a certain 4-dimensional abelian variety. Amer. J. Math. 108 (1986), no. 4, 931–952.