Topology Qualifying Exam
August 2004

1. Let X be a non-compact locally compact Hausdorff space, with topology $\mathcal{T}$. Let $\widehat{X} := X \cup \{\infty\}$ (X with one point adjoined), and consider the family $\mathcal{B}$ of subsets of $\widehat{X}$ defined by

$$\mathcal{B} := \mathcal{T} \cup \{S \cup \{\infty\} : S \subset X,\ X - S \text{ compact}\}.$$

a) Prove that $\mathcal{B}$ is a topology on $\widehat{X}$, that the resulting space is compact, and that X is dense in $\widehat{X}$.

b) Prove that if $Y \supset X$ is a compact space such that X is dense in Y and $Y - X$ is a singleton, then Y is homeomorphic to $\widehat{X}$. (The space $\widehat{X}$ is called the *one-point compactification* of X.)

c) Find the one-point compactifications of i) $X = (0, 1)$ and ii) $X = \mathbb{R}^2$.

2. Let X be a compact Hausdorff space and suppose $R \subset X \times X$ is a closed equivalence relation. Show that the quotient space X/R is Hausdorff.

3. Let X be a topological space.

a) Prove that X is connected if and only if there is no continuous nonconstant map to the discrete two-point space $\{0, 1\}$.

b) Suppose in addition that X is compact and that Y is a connected Hausdorff space. Suppose further that there is a surjective continuous map $f : X \to Y$ with the property that every pre-image $f^{-1}(y)$, $y \in Y$, is a connected subspace of X. Show that X is connected.

c) Give an example showing that the conclusion of b) may be false if X is not compact.

4. Let X be the space formed by identifying the boundary of a Möbius band M with a meridian of the torus T^2. Compute $\pi_1(X)$ and $H_*(X)$.

5. Describe the 3-fold connected covering spaces of $S^1 \vee S^1$.

6. Let M_g^2 be the compact oriented surface of genus g. Show that there exists a continuous map $f : M_g^2 \to S^2$ that is not homotopic to a constant map.

7. Let $f, g : S^n \to S^n$, where $f(x) \neq g(x)$ for all x. Show that f is homotopic to $a \circ g$, where $a(x) = -x$ is the antipodal map.

8. Let $f : S^{2n} \to S^{2n}$. Show that there exists at least one point x such that $f(x) = \pm x$.