THE UNCERTAINTY PRINCIPLE

1. Definitions

Definition 1.1. *The Fourier coefficients of a periodic function with period* 1 *are defined by*

$$\hat{f}(n) = \int_0^1 f(x)e^{-2\pi inx}dx$$

for $n \in \mathbb{Z}$. *Since* $\int_0^1 e^{-2\pi inx}dx = 0$ *for all* $n \in \mathbb{Z}\backslash\{0\}$, *we have for* $f(x) = e^{2\pi ikx}$

$$\hat{f}(n) = \int_0^1 e^{-2\pi i(n-k)x}dx = \begin{cases} 0 & \text{if} \quad n \neq k \\ 1 & \text{if} \quad n = k \end{cases}$$

Then $\hat{f}(n)$ *measures how much the function* f *vibrates at frequency* n. *Moreover, using the synthesis formula* $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{2\pi inx}$, *one can see the function* f *as composed of functions vibrating at different frequencies each with amplitude* $\hat{f}(n)$.

Definition 1.2. *For* $\xi \in \mathbb{R}$, *Fourier transform of an integrable function on* $\mathbb{R}$ *is defined by*

$$\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(x)e^{-2\pi i\xi x}dx$$

The Fourier transform satisfies the following properties:

- The Fourier transform relates the translation and modulation operators $\mathcal{T}_a f(x) = f(x-a)$, $\mathcal{M}_a f(x) = e^{2\pi iax}f(x)$ as follows: $\widehat{\mathcal{T}_a f} = \mathcal{M}_{-a}\widehat{f}$ and $\widehat{\mathcal{M}_a f} = \mathcal{T}_a\widehat{f}$
- The Fourier transform relates derivatives with products: $\widehat{f'}(\xi) = 2\pi i\xi\widehat{f}(\xi)$
- Inversion formula: $f(x) = \int \widehat{f}(\xi)e^{2\pi ix\xi}d\xi$
- Plancharel's identity: $\langle \hat{f}, \hat{g}\rangle = \langle f, g\rangle$ or $\int \hat{f}(x)\overline{\hat{g}(x)}dx = \int f(x)\overline{g(x)}dx$
- Parseval's identity (generalisation of the Pythagorean Theorem): if we denote $\|f\|_2 = \left(\int |f(x)|^2dx\right)^{1/2} = \langle f, f\rangle^{1/2}$, then $\|\hat{f}\|_2 = \|f\|_2$.

Finally, the Cauchy-Schwarz's inequality is a tool frequently used in Analysis stating that

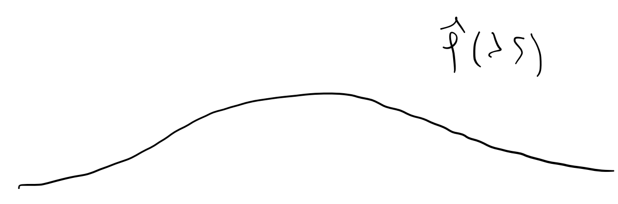
$$\left|\int f(x)g(x)dx\right| \leq \left(\int |f(x)|^2dx\right)^{\frac{1}{2}}\left(\int |g(x)|^2dx\right)^{\frac{1}{2}}.$$

It can also be written as $|\langle f, g\rangle| \leq \|f\|_2\|g\|_2$.

2. The Uncertainty Principle in Harmonic Analysis

In Harmonic Analysis, the uncertainty principle can be succinctly stated as follows: a nonzero function and its Fourier transform cannot both be sharply localised. That is, if a function is restricted to a narrow region of the physical space, then its Fourier transform must spread (be essentially constant) over a broad region of the frequency space. It then expresses a limitation on the extent to which a signal can be both time-limited and band-limited.

The most simple way to write mathematically this property is by means of dilations: if we define the dilation operator $\mathcal{D}_\lambda f(x) = \lambda^{-1}f(\lambda^{-1}x)$, then $\widehat{\mathcal{D}_\lambda f}(\xi) = \widehat{f}(\lambda\xi)$.



In quantum mechanics, the uncertainty principle is stated in the following way: the position and momentum of a particle cannot be measured simultaneously. To be able to talk about accuracy of measurements, we need two concepts from Probability Theory: the expectation and the standard deviation.

Let f be a measurable function such that $|f|^2$ is a density probability function ($\int |f(x)|^2 dx = 1$). The expectation is defined by $\langle x\rangle = E(|f|^2) = \int x|f(x)|^2 dx$ and the standard deviation is defined as

$$\sigma_x^2 = V(|f|^2) = \int (x-\langle x\rangle)^2 |f(x)|^2 dx = \int (x^2 - \langle x\rangle^2)|f(x)|^2 dx = \inf_a \int (x-a)^2 |fx)|^2 dx$$

Then Heisenberg's uncertainty principle can be mathematically expressed as $\sigma_x \cdot \sigma_p \geq \frac{\hbar}{2}$.

In this talk, as it is customary in Harmonic Analysis, the constant is normalised so that $\hbar = \frac{1}{2\pi}$.

Theorem 2.1. *(Heisenberg's uncertainty principle I) Let f be a measurable function with $\|f\|_2 = 1$. Then $V(|f|^2)V(|\widehat{f}|^2) \geq (4\pi)^{-2}$. In fact, for all $x_0, \xi_0 \in \mathbb{R}$,*

$$\int (x-x_0)^2 |f(x)|^2 dx \int (\xi - \xi_0)^2 |\widehat{f}(\xi)|^2 d\xi \geq (4\pi)^{-2}$$

Proof. Changing f by $T_{-x_0}M_{-\xi_0}f$, the inequality is proved once we show

$$\int x^2 |f(x)|^2 dx \int \xi^2 |\widehat{f}(\xi)|^2 d\xi \geq (4\pi)^{-2}.$$

Using integration by parts, we have

$$\begin{aligned}1 = \int |f(x)|^2 dx &= \int \frac{d}{dx}(x) f(x)\overline{f(x)} dx = -\int x \frac{d}{dx}(f\overline{f})(x) dx \\ &= -\int x(f'(x)\overline{f(x)} + f(x)\overline{f'(x)})dx = -2\operatorname{Re}\Big(\int x f(x)\overline{f'(x)} dx\Big)\end{aligned}$$

Now, by Cauchy-Schwarz inequality

$$1 \leq 2\int |x||f(x)||f'(x)| dx \leq 2\Big(\int x^2 |f(x)|^2 dx\Big)^{\frac{1}{2}} \Big(\int |f'(\xi)|^2 d\xi\Big)^{\frac{1}{2}}$$

Using $\widehat{f'}(\xi) = 2\pi i \xi \hat{f}(\xi)$ and Parseval's identity, we have

$$\Big(\int |f'(x)|^2 dx\Big)^{\frac{1}{2}} = \|f'\|_2 = \|\widehat{f'}\|_2 = \|2\pi i \cdot \widehat{f}\|_2 = 2\pi \Big(\int \xi^2 |\widehat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}$$

and so,

$$1 \leq 4\pi \Big(\int x^2 |f(x)|^2 dx\Big)^{\frac{1}{2}} \Big(\int \xi^2 |\widehat{f}(\xi)|^2 d\xi\Big)^{\frac{1}{2}}$$

$\square$

To interpret this theorem, we assume that the position of a particle is given by the density probability function $|f|^2$: the probability that the particle is located in the interval (a,b) is $\int_a^b |f(x)|^2 dx$. Then the expected value $\langle x\rangle = E(|f|^2)$ is the averaged position (an observable, which can be actually measured) while the standard deviation $\sigma_x = V(|f|^2)^{1/2}$ measures the average deviation from the expected value, that is, a measurement of the uncertainty attached to the expectation.

Moreover, we also assume that the probability that the momentum of a particle belongs to the interval (a,b) is $\int_a^b |\widehat{f}(\xi)|^2 d\xi$. The rationale behind this assumption is the duality between the position operator $xf(x)$ and the derivative operator $(2\pi i)^{-1} f'(x)$, as we explain. While $\langle x\rangle = \int x f(x)\overline{f(x)} dx$, we have

$$\langle p\rangle = \frac{1}{2\pi i}\int f'(x)\overline{f(x)} dx = \frac{1}{2\pi i}\int \widehat{f'}(\xi)\overline{\widehat{f}(\xi)} d\xi = \int \xi \widehat{f}(\xi)\overline{\widehat{f}(\xi)} d\xi = \int \xi |\widehat{f}(\xi)|^2 d\xi$$

and so, $|\widehat{f}(\xi)|^2$ is the probability density for the momentum. Then, while $\sigma_x^2 = \int x^2|f(x)|^2dx$, we have

$$\sigma_p^2 = \int \xi^2|\widehat{f}(\xi)|^2dx = \frac{1}{2\pi i}\int |f'(x)|^2dx$$

The result does not imply that a particle does not have well-defined position and momentum. At any time, any of these quantities can be measured with arbitrary precision with the appropriate approximation of the identity. Using $f(x) = \epsilon^{-1}\varphi(\epsilon x)$ we can determine the position but it produces uncertainty in the momentum. Using $f(x) = \tilde{\varphi}(\epsilon^{-1}x)$ instead, allows to determine the momentum but implies lack of certainty for the position. That is, measuring one of the variables interferes with the measurement of the other one.

3. The Uncertainty Principle in Operator Theory

We define the operators $Xf(x) = xf(x)$ and $Df(x) = \frac{1}{2\pi i}f'(x)$. From the preceding discussion we have $\widehat{Df} = X\widehat{f}$, that is, X multiplies by the physical variable, while D multiplies by the frequency variable. The quantity

$$\sigma_x = \frac{\|Xf\|_2}{\|f\|_2} = \left(\frac{\int x^2|f(x)|^2dt}{\int |f(x)|^2dt}\right)^{\frac{1}{2}}$$

is a measure of the average value of $|t|$, or in other words the average deviation of the physical variable x from the origin. Similarly

$$\sigma_p = \frac{\|Df\|_2}{\|f\|_2} = \frac{\|\widehat{Df}\|_2}{\|\widehat{f}\|_2} = \left(\frac{\int \xi^2|\widehat{f}(\xi)|^2d\tau}{\int |\widehat{f}(\xi)|^2d\tau}\right)^{\frac{1}{2}}$$

measures the average deviation of the frequency variable ξ from the origin. Note that X and D are both self-adjoint operators and that they do not commute: on the one side $XDf(x) = xDf(x)$ while, on the other side, $DXf(x) = D(xf(x)) = \frac{1}{2\pi i}f(x) + xDf(x)$. Then,

$$[X, D]f = XDf - DXf = -\frac{1}{2\pi i}f$$

Proposition 3.1. *(Heisenberg uncertainty principle II) We have*

$$\frac{\|Xf\|_2}{\|f\|_2}\frac{\|Df\|_2}{\|f\|_2} \geq \frac{1}{4\pi}$$

Proof. We consider the quantity $\|(aX + ibD)f\|_2$, with a, b real numbers to be chosen later. Clearly this quantity is non-negative. Moreover,

$$\begin{aligned}
0 \leq \|(aX + ibD)f\|_2 &= \langle (aX + ibD)f, (aX + ibD)f\rangle \\
&= a^2\langle Xf, Xf\rangle + b^2\langle Df, Df\rangle + iab\langle Xf, Df\rangle - iab\langle Df, Xf\rangle \\
&= a^2\langle Xf, Xf\rangle + b^2\langle Df, Df\rangle + iab\left(\langle DXf, f\rangle - \langle XDf, f\rangle\right) \\
&= a^2\|Xf\|_2^2 + b^2\|Df\|_2^2 - iab\langle [X, D]f, f\rangle \\
&= a^2\|Xf\|_2^2 + b^2\|Df\|_2^2 - \frac{ab}{2\pi}\|f\|_2^2.
\end{aligned}$$

Then we have

$$\frac{a}{b}\|Xf\|_2^2 + \frac{b}{a}\|Df\|_2^2 \geq \frac{1}{2\pi}\|f\|_2^2.$$

Now, the function $f(\lambda) = \lambda u^2 + \lambda^{-1}v^2$ attains its minimum at $\lambda = \frac{v}{u}(= \frac{a}{b})$, for which $f(\frac{v}{u}) = 2uv$. For that reason, we pick $a = v = \|Df\|_2$ and $b = u = \|Xf\|_2$, and we thus have

$$2\|Xf\|_2\|Df\|_2 \geq \frac{1}{2\pi}\|f\|_2^2$$

□

The uncertainty principle also applies to other pairs of operators. For instance, the operators energy and time: if time is measured with high precision, energy cannot be accurately measured. This phenomenon implies that during a small interval of time, the principle of conservation of energy can be violated. This property gives an explanation for the existence of virtual particles, which appear spontaneously in the vacuum as a pair of particle and anti-particle to annihilate each other shortly after. It also explains Hawking's radiation of black holes and it ultimately provides a speculative explanation for the origin of the Universe out of non-existence.

In the previous proof, we essentially used self-adjointness of the operators and their lack of commutativity. So, the question is: do all self-adjoint operators that do not commute satisfy a similar inequality? The answer is... almost.

Let A, B be two operators densely defined on a Hilbert space $\mathcal{H}$ with domains $D(A)$ and $D(B)$. Then the domain of the product is $D(AB) = \{u \in D(B) : Bu \in D(A)\}$ and similar for $D(BA)$. The commutator is defined on $D([A,B]) = D(AB) \cap D(BA) \subset D(A) \cap D(B)$.

Proposition 3.2. *Let A, B be self-adjoint operators and $\alpha, \beta \in \mathbb{C}$. Then*

$$\|(A-\alpha I)u\| \|(B-\beta I)u\| \geq \frac{1}{2} |\langle [A,B]u, u\rangle|$$

for all $u \in D([A,B])$.

Proof. We denote $\tilde{A} = A - \alpha I$ and $\tilde{B} = B - \beta I$. Since $[A,B] = [A-\alpha I, B - \beta I]$, we have,

$$\begin{aligned}|\langle [A,B]u,u\rangle| &= |\langle [\tilde{A},\tilde{B}]u,u\rangle| = |\langle \tilde{A}\tilde{B}u,u\rangle - \langle \tilde{B}\tilde{A}u,u\rangle| \\ &= |\langle \tilde{B}u,\tilde{A}u\rangle - \langle \tilde{A}u,\tilde{B}u\rangle| = 2|\operatorname{Im}\langle \tilde{A}u,\tilde{B}u\rangle| \leq 2\|\tilde{A}u\|\|\tilde{B}u\|\end{aligned}$$

□

An important remark about this last result should be added. Its simple statement and easy proof hide some subtleties that prevent answering previous question in the positive. Not all all self-adjoint operators that do not commute satisfy the inequality. On the one hand, $D([A,B])$ could be non-dense or could even be empty, although this is not typically the case. More importantly, $[A,B]$ is not in general a closed operator. Even denoting its closure by $C = \operatorname{cl}[A,B]$, it turns out that the inequality

$$\|Au\|\|Bu\| \geq \frac{1}{2}|\langle Cu,u\rangle|$$

for all $u \in D(A) \cap D(B) \cap D(C)$ is false in general. For that reason, refinements of this result are still a subject of research in Operator Theory.

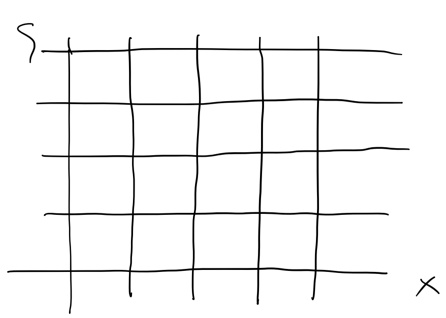
4. Bases of L^2

The uncertainty principle needs to be taken into account (consciously or unconscioulsy) when constructing bases or frames for L^2. The idea is the following one: suppose a function f is concentrated on an interval I while $\widehat{f}$ is concentrated on an interval I'. In that case, we can say that f is represented by the rectangle $I \times I'$. Then, by the uncertainty principle, we must have $|I||I'| \geq 1$. This way, in order to construct a complete system of functions for L^2, $(\psi_{j,k})_{j,k}$, such that each function is represented by a rectangle $I_j \times I_k$, by the uncertainty principle, we need to guarantee first that $|I_j||I_k| \geq 1$. This is the case of the Gabor system

$$\phi_{j,k}(x) = \varphi(x-j)e^{2\pi i k x} = \mathcal{T}_j \mathcal{M}_k \varphi(x)$$

and the wavelet system

$$\psi_{j,k}(x) = 2^{-k/2}\varphi(2^{-k}x - j) = \mathcal{T}_{j2^k}\mathcal{D}_{2^k,2}\varphi(x)$$



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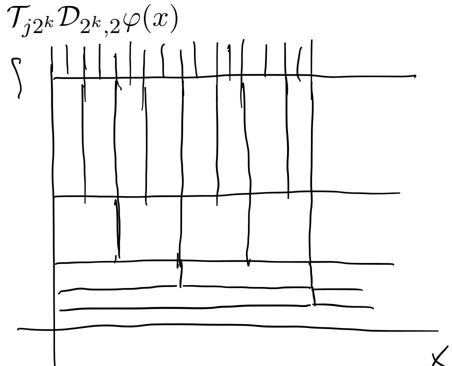

WAVELETS

5. Other forms of the uncertainty principle

The following results are other different ways of formulating the uncertainty principle.

Theorem 5.1. *(Hardy's uncertainty principle) Let $a, b > 0$. Let f such that $|f(x)| \leq Ce^{-a\pi x^2}$ and $|\widehat{f}(\xi)| \leq Ce^{-b\pi\xi^2}$ for some $C > 0$. If $ab > 1$, then $f \equiv 0$. If $ab = 1$, then $f(x) = Ae^{-a\pi x^2}$.*

Theorem 5.2. *(Beurling's uncertainty principle) If*

$$\int\int |f(x)||\widehat{f}(\xi)|e^{-a\pi x\xi}dxd\xi < \infty$$

then $f \equiv 0$.

Theorem 5.3. *(Amrein-Berthier's uncertainty principle) There is $C > 0$ such that for all $A, B \subset \mathbb{R}$ of finite measure and all $f \in L^2$,*

$$\|f\|_2 \leq Ce^{C|A||B|}\Big(\int_{\mathbb{R}\setminus A} |f(x)|^2dx + \int_{\mathbb{R}\setminus B} |\widehat{f}(\xi)|^2d\xi\Big)$$

In particular, if both f and $\widehat{f}$ are supported on sets of finite measure, then $f \equiv 0$.

Theorem 5.4. *(Poisson Summation Formula) Let f be an integrable function. Then*

$$\sum_{k\in\mathbb{Z}} f(x+k) = \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{2\pi ikx}$$

Theorem 5.5. *(Bernstein's Theorem) Assume that $f \in L^2$ and $\widehat{f}$ is supported in $D(0, R)$. Then f is C^∞ and the following inequalities hold:*

1. *For any α and $1 \leq p \leq \infty$, $\|D^\alpha f\|_p \leq (2\pi R)^{|\alpha|}\|f\|_p$*
2. *For any $1 \leq p \leq q \leq \infty$, $\|f\|_q \leq CR^{\frac{1}{p}-\frac{1}{q}}\|f\|_p$*

Theorem 5.6. *(Paley-Wiener's Theorem) Let $f \in C_0^\infty(\mathbb{R})$, and suppose $\widehat{f}(\xi) = 0$ for $|\xi| > R$. Then f can be extended to an entire function on $\mathbb{C}$ with the following decay estimate*

$$|f(z)| \leq C_N(1 + |z|)^N e^{R|\mathrm{Im}z|}$$

for any $N \in \mathbb{Z}^+$.

Theorem 5.7. *(Montgomery's Uncertainty Principle) Let $f : \mathbb{Z} \to \mathbb{C}$ be a finitely supported function which, for each prime p, avoids $\omega(p)$ residue classes modulo p for some $0 \leq \omega(p) < p$. Then for each natural number q,*

$$\sum_{\substack{1 \leq a \leq q \\ (a,p)=1}} |\widehat{f}(\xi + \frac{a}{q})|^2 \geq C_q|\widehat{f}(\xi)|^2$$

where $C_q = \mu(q)^2 \prod_{p|q} \frac{\omega(q)}{p-\omega(p)}$ and μ is the Mobius function.

The result means that the more classes a function excludes, the more Fourier energy has to disperse along multiples of $1/p$. This is used to bound the measure of sets that avoid many module classes and to show that certain sets must contain arithmetic progression of certain lengths.