Evasive Random Walks

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The Clairvoyant Demon

We will consider the following beautiful problem from a paper of Coppersmith, Tetali, and Winkler ([1]), which they call the "clairvoyant demon" problem. Let $X = 0.x_0x_1x_2\ldots$ and $Y = 0.y_0y_1y_2\ldots$ be the decimal expansions of two real numbers chosen uniformly at random from the interval $[0,1]$. Suppose that at time 0 you (the demon) have two tokens, an X-token sitting on the digit x_0 and a Y-token sitting on y_0. At each time step you must move either the X-token or the Y-token one digit to the right; that is, if at time t the tokens are at x_i and y_j then at time $t+1$ they will either be at x_{i+1} and y_j or x_i and y_{j+1}. (Note that here $t = i + j$.) Your *goal* as demon is to move both tokens infinitely many times. Your *constraint* is key: the two tokens may never sit on the same number.

Now, depending on X and Y, it may or may not be possible for you to succeed. For instance, if $X = 0.1111\ldots$ and $Y = 0.3333\ldots$ then it is clearly possible; we say that you *win* with (X,Y). On the other hand if $X = 0.46\ldots$ and $Y = 0.64\ldots$ then it is impossible, since you cannot make a single move. Here you *lose* with (X,Y). Note that if you lose with (X,Y), then this can be detected by looking at finite initial strings of X and Y.

The problem, then, is to determine whether you have a positive chance of winning.

Other formulations

There are (at least) two other attractive ways to think about this problem. First, we may view X and Y as random walks on a graph, namely the complete graph on the ten vertices $0, 1, \ldots, 9$. The random walk X starts at the vertex labelled x_0, then proceeds to x_1, etc. [There is a slight technical issue of "repeated digits" in the translation.] Simultaneously thinking of X and Y as the paths of the X-token and the Y-token, respectively, we may rephrase the question as

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follows: Is it possible to parameterize X and Y (in discrete time) so that they never collide? I.e., can the demon keep X and Y apart forever?

This becomes a question about (finite) graphs. Namely, if G is a graph, let P_G denote the probability that you can win with two random walks on G. Then for what G is $P_G > 0$? No graph is known to have $P_G > 0$.

Conjecture 1 (Winkler) *$P_G > 0$ if $G = K_4$.*

It is known [Winkler, personal communication] that if G is a cycle then $P_G = 0$. We have obtained the following bound.

Theorem 1 *Let G be a cycle of length n, and let X and Y be random walks on G. Then by looking at initial strings of X and Y of length Nn, we can detect that the demon loses with (X, Y) with probability at least*

$$1 - e^{c\sqrt[3]{N}},$$

where c is some (negative) constant.

The problem can also be formulated in the language of percolation. Let us pay attention to the locations of the tokens as they move along the strings X and Y. We say the tokens are at (i, j) if the X-token is on x_i and the Y-token is on y_j. Then we see a north-and-east-ward moving path through $\mathbb{N} \times \mathbb{N}$, beginning at $(0, 0)$. The path must avoid certain "bad" sites, namely $\{(i, j) :: x_i = y_j\}$ (see Figure 1). The problem is whether, with positive probability, you can connect $(0, 0)$ to infinity in this "directed" percolation setting. Notice that there is some dependence among the sites, so standard results about independent percolation don't apply.

Winkler and independently Balister, Bollobás, and Stacey have recently shown ([3], [4]) that if the demon is allowed to also move backward along the paths X and Y, so that the percolation is undirected (but still dependent), then the demon has a positive chance of winning on K_4.

An Approach

We have investigated the following variant. Fix the graph G. Is it possible to choose X ahead of time so that if Y is then chosen randomly, the probability of winning with (X, Y) is positive? Such an X we will call *evasive*. If there are no evasive walks on G then $P_G = 0$; if the set of evasive walks on G has positive measure then $P_G > 0$. We have shown:

Theorem 2 *There exist uncountably many evasive walks on K_4.*

This particular set of evasive walks has measure 0, however, because our construction produces walks X with the property that some vertex of K_4 appears in X much more sparsely than the rest. Our next result begins to resolve this difficulty. We say that a walk X on G visits the vertex v of G with *upper density* d if

$$\limsup_{k\to\infty} \frac{\#\{i :: 0 \le i \le k \text{ and } x_i = v\}}{k+1} = d.$$

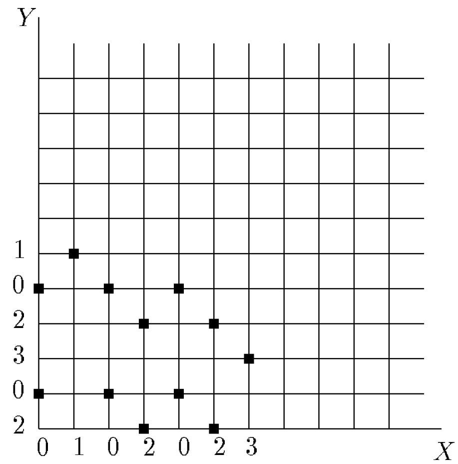

Figure 1: Percolation. The bad sites are indicated with black squares. The demon loses if X begins $0102023\ldots$ and Y begins $203201\ldots$.

Theorem 3 *If $n \geq 5$, there exist uncountably many evasive walks on K_n, each of which visits every vertex with upper density at least $1/n$.*

This is progress, but this set of evasive walks also seems to have measure 0. So far there is no graph on which we can construct a set of evasive walks which we can prove has positive measure. We remain hopeful, though, that our ideas can be extended and our results improved.

References

[1] D. Coppersmith, P. Tetali, and P. Winkler, *Collisions Among Random Walks on a Graph*, SIAM J. Discrete Math 6 (1993), no. 3, 363–374.

[2] A. Abrams, H. Landau, Z. Landau, J. Pommersheim, and E. Zaslow, *The Clairvoyant Demon and Related Problems*, in preparation.

[3] P. Winkler, *Dependent Percolation and Colliding Random Walks*, preprint.

[4] P. N. Balister, B. Bollobás, and A. N. Stacey, *Dependent Percolation in Two Dimensions*, preprint.