Strategies for a Sequenced Game of Chance

Aja Johnson*

January 11, 2005

Abstract

In a game of chance wherein two players spin a uniformly distributed n-sectioned wheel with the opportunity to spin twice, the winner is the person with the highest sum that is not over n. For each value of n, there is a unique cut-off point where Player One should switch from spinning again to staying.

Introduction

At the end of the game show The Price Is Right, players have a spin-off to see who gets to move on to the final round. The three players spin a wheel (with the opportunity to spin twice) and the winner is the person with the highest score that is not over one dollar. There are 20 sections on the wheel that are numbered in multiples of five (i.e. 5 cents, 10 cents, etc.). The chance of spinning each number is equally likely. What are the players' strategies for winning?

First Step

Begin with a variation of this problem; these variables can be changed later for further research. The assumptions:

1. There are two players, Player One and Player Two.
2. There is one roulette wheel with equally likely values 1 through n.
3. Player One spins first, with a choice to spin once more.
4. Player Two spins after Player One, with a choice to spin once more.
5. Player One wins if and only if Player One's total spin sum is not greater than n, and Player Two's total spin sum is
   (a) greater than n, or
   (b) less than or equal to Player One's spin sum.

---

*Elon University

The second player's strategy is simple. If she beats the first player on the first spin, then she should not spin again; she has won. If she does not beat the first player on her first spin, then she should spin again because this is the only way that she can possibly win. There is not such a clear-cut rule for the first player to follow.

The first player only wants to spin again if it increases her chance of winning. The ways that the first player can win are if the second player's sum does not beat her sum, or if the second player exceeds n. While spinning again raises the probability of Player One getting a higher sum, it also raises the probability of her exceeding n. How does she decide if it is worth the risk?

How Player Two Can Lose

Since Player Two's strategy is to spin until she wins, the only way that she can lose is if she does not beat Player One on her first spin and on her second spin.

Let's first look at the probability that Player Two does not beat Player One on her first spin. Let T_j be the sum of Player j's spins. Player Two loses on her first spin if she gets any number less than or equal to T_1, so the total number of ways that Player Two can lose on her first spin is T_1. She has a $\frac{1}{n}$ probability of spinning each of the values from 1 to T_1, so the probability that she spins one of these values and thus loses is $\dfrac{T_1}{n}$.

To find the number of ways that Player Two can lose on her second spin consider the number of ways that she could win. Regardless of the value of her first spin, there are only $n - T_1$ numbers that would cause her to win overall, so there are only $n - T_1$ numbers that could be added to her first spin that would cause her to win. Therefore the number of ways that she can lose is the number of possible spins less the number of ways that she can win, $n - (n - T_1) = T_1$. Again the probability of spinning one of these values is $\dfrac{T_1}{n}$. Thus, the probability that Player Two loses is

$$\frac{T_1}{n} \cdot \frac{T_1}{n} = \left(\frac{T_1}{n}\right)^2.$$

Example. *Let's say that Player One spins a wheel of size 5, gets a 3 on her first spin, and stops; thus her total is 3. The probability of Player Two losing is*

$$\frac{3^2}{5^2} = \frac{9}{25} = .36.$$

Player One's Strategy

The probability of Player Two losing equals the probability of Player One winning because there are only two players. Let $r(j,k)$ represent the kth spin of the jth player. The results that we have just found can be considered the probability of Player One winning given that she only spins once. This means that $r(1,1) = T_1$ and $r(1,2) = 0$.

Now consider the probability of Player One winning given that she spins again. Look at this in terms of the expression that we just found. Let $r(1,1) = u$, $r(1,2) = v$, and thus $T_1 = u + v$. The probability that Player One wins given that $v = 0$ is $\left(\frac{(u+0)}{n}\right)^2$. The probability of her winning if $v \neq 0$ can be found by summing over all the values of v that do not cause her to exceed n, the probability of spinning v times the probability that the resulting total will cause her to win. This is represented by

$$\begin{aligned}
\sum_{v=1}^{n-u}\left(\frac{1}{n}\right)\left(\frac{(u+v)}{n}\right)^2 &= \frac{1}{n^3}\sum_{v=1}^{n-u}(u+v)^2,\\
&= \frac{1}{n^3}\sum_{T_1=u+1}^{n} T_1^2,\\
&= \frac{1}{n^3}\left(\sum_{T_1=1}^{n}, T_1^2 - \sum_{T_1=1}^{u} T_1^2\right)\\
&= \frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - \frac{u(u+1)(2u+1)}{6}\right),\\
&= \frac{1}{6n^3}(n(n+1)(2n+1) - u(u+1)(2u+1)).
\end{aligned}$$

Let this cubic be $g_2(u)$. This is Player One's probability of winning given that $r(1,1) = u$ and $r(1,2) \neq 0$.

Example. *Let's again look at the instance where Player One spins a 3 on her first spin on a wheel of size 5. The probability of her winning if she spins again is*

$$\frac{1}{6\cdot 5^3}(5(5+1)(2\cdot 5+1) - 3(3+1)(2\cdot 3+1)) = \frac{41}{125} = .328.$$

Note that $.36 > .328$, *so Player One is slightly more likely to win if she stays with a 3 than if she spins again.*

Letting $g_1(u) = \frac{u^2}{n^2}$, compare the graphs of $g_1(u)$ and $g_2(u)$ (Figure 1). If and where these curves cross is where Player One should switch from spinning to staying or vice versa.

The bold entries in Table 1 indicate the better probability for winning. Both Table 1 and Figure 1 illustrate the crossover mark for $n = 5$ to be somewhere between 2 and 3. This means that Player One's strategy should be to spin again when $r(1,1) \leq 2$, and to stay when $r(1,1) \geq 3$.

The intersections of the two curves can be found by solving $g_1(u) = g_2(u)$.

$$\begin{aligned}
0 &= g_1(u) - g_2(u)\\
&= \frac{2u^3 + (3+6n)u^2 + u}{n^3} - \frac{n(n+1)(2n+1)}{n^3}\\
&= 2u^3 + (3+6n)u^2 + u - n(n+1)(2n+1).
\end{aligned}$$

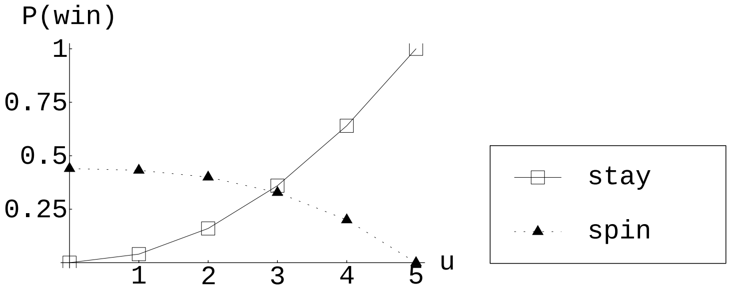

Figure 1: A discrete graph of the probability of winning for staying and spinning when $n = 5$.

Let $g(u)$ be this cubic.

Player One's decision to stay or spin changes at the roots of $g(u)$. How many roots are there of this cubic? More importantly how many positive roots are there?

$r(1,1)$	P(winning when $r(1,2) = 0$)	P(winning when $r(1,2) \neq 0$)
1	$\frac{1}{25} = .04$	$\mathbf{\frac{54}{125} = .432}$
2	$\frac{4}{25} = .16$	$\mathbf{\frac{50}{125} = .4}$
3	$\mathbf{\frac{9}{25} = .36}$	$\frac{41}{125} = .328$
4	$\mathbf{\frac{16}{25} = .64}$	$\frac{25}{125} = .2$
5	**1**	0

Table 1: How Player One can win for $n = 5$.

Theorem 1. *Under the assumptions, there is exactly one positive root of*

$$g(u) = 2u^3 + (3 + 6n)u^2 + u - n(n + 1)(2n + 1).$$

Proof. Evaluating $g(u)$ at $u = 0$ and $u = n$ yields:

$$g(0) = -n(n + 1)(2n + 1) < 0,$$

$$g(n) = 2n^3 + (3 + 6n)n^2 + n - n(n + 1)(2n + 1) = 6n^3 > 0.$$

For all n, $g(0) < 0$ and $g(n) > 0$. By the intermediate value theorem, since $g(u)$ is continuous, there must exist a root of $g(u)$ between 0 and n.

Exactly how many positive roots does this polynomial have? The first derivative of $g(u)$ is

$$g'(u) = 6u^2 + 2(3 + 6n)u + 1.$$

Note that $g'(u)$ is positive for $u \geq 0$, thus $g(u)$ is increasing over this domain. $g(u)$ can cross the u-axis at most once for values $u \geq 0$, and therefore, has exactly one positive root. $\square$

Call this unique positive root c_n. The strategy of Player One is defined by c_n.

Player One's Strategy. *If on the first spin, Player One gets a value $u < c_n$, Player One will spin again. If $u > c_n$, then Player One stays. If $u = c_n$ then Player One can choose to spin again or stay. The unique root c_n is Player One's strategic cut-off value for a wheel of size n.*

What is this exact root? Since $g(u)$ is a cubic, this question can be answered exactly using Mathematica:

$$c_n = \frac{-1 + \left(-b - i\sqrt{a^3 - b^2}\right)^{\frac{1}{3}} + \left(-b + i\sqrt{a^3 - b^2}\right)^{\frac{1}{3}} - 2\,n}{2},$$

where $a = -\left(\frac{2}{3}\right) + (1 + 2\,n)^2$, and $b = 2\,n\;(1 + n)\;(1 + 2\,n)$.

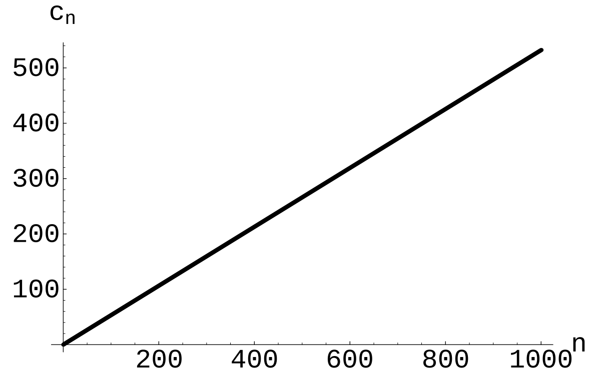

Figure 2: A graph of the values of n versus c_n.

Figure 2 shows a graph of n versus c_n. The linear nature of this graph implies that c_n will always be approximately a fixed fraction of n. In Table 2 we can see that each value of c_n is slightly more than $\frac{n}{2}$. Figure 3 shows a graph of n versus $\frac{c_n}{n}$. It appears that this curve is asymptotic to some line as it approaches infinity. What is this line?

n	2	5	10	15	20	25
cut-off point, c_n	1.27898	2.90326	5.57477	8.23914	10.90160	13.56330
$\frac{c_n}{n}$	.63949	.58065	.55748	.54928	.54508	.54253

Table 2: Cut-off points for selected n values.

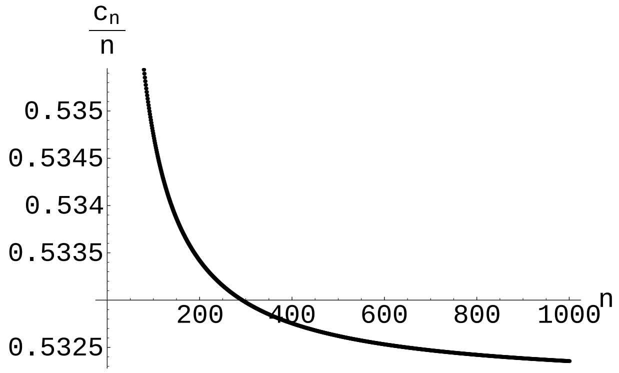

Figure 3: A graph of the values of n versus $\frac{c_n}{n}$.

Theorem 2. *The horizontal asymptote of Figure 3 is* $y = -1 + 2\cos(\frac{2\pi}{9})$.

Proof. We evaluate for the limit of our curve as it approaches infinity and see that

$$\lim_{n\to\infty} \frac{c_n}{n} = -1 + \frac{1}{(\frac{1}{2}i(i+\sqrt{3}))^{\frac{1}{3}}} + (\frac{1}{2}i(i+\sqrt{3}))^{\frac{1}{3}} = -1 + 2\cos(\frac{2\pi}{9}) \approx .532089.$$

□

This means that the cut-off value will occur at approximately 53% of the value of n for large n. This finding agrees with what is seen in Figure 3 and Table 2.

Bounding the Cut-Off Value

As seen earlier, $0 < c_n < n$. Can better bounds be found for c_n?

Theorem 3. *The root* c_n *can always be bounded in the following way:*

$$\frac{n}{2} < c_n < \frac{8\,n}{15} + \frac{43}{150}.$$

Proof. Look at what happens when $u = \dfrac{n}{2}$,

$$g\left(\frac{n}{2}\right) = 2\left(\frac{n}{2}\right)^3 + (3+6n)\left(\frac{n}{2}\right)^2 + \left(\frac{n}{2}\right) - n(n+1)(2n+1) = -\frac{n}{2} - \frac{n^2}{4} - \frac{n^3}{4}.$$

Since $g\left(\dfrac{n}{2}\right) < 0$, this becomes a lower bound. This is a very interesting result, considering Player One might just assume that she should switch from spinning to staying halfway between 0 and n. This is a reasonable strategy for wheel with large n values, since the cut-off value is approximately half of the value of n. On wheels with small n values, this strategy could cause a small but significant drop in the probability that Player One wins.

Another way to find better bounds for the value of c_n is to use Newton's method for approximation. The second derivative of the function $g(u)$ is

$$g''(u) = 12u + (6 + 12n).$$

Note that $g''(u)$ is always positive for $u \geq 0$, so $g(u)$ is concave up over this domain. Consider the line $y = g'(u_0)(u - u_0) + g(u_o)$ tangent to $g(u)$ at $(u_0, g(u_0))$. Since the graph of $g(u)$ is concave up, then tangent lines lie below the graph, thus the root of a tangent line must lie to the right of the root c_n.

$$c_n < l = u_0 - \frac{g(u_0)}{g'(u_0)}, \text{ for } c_n \neq u_0.$$

Evaluating for when $u_0 = \frac{n}{2}$, yields

$$c_n < l = \frac{n(4 + 15n + 16n^2)}{4 + 12n + 30n^2} = \frac{8n}{15} + \frac{43}{150} - \frac{59n + 43}{75(2 + 6n + 15n^2)}.$$

This value will always be less than $\dfrac{8n}{15} + \dfrac{43}{150}$ when $n > 1$. □

The fact that $\dfrac{8n}{15} + \dfrac{43}{150}$ is an upper bound for c_n gives that $\dfrac{8}{15} + \dfrac{43}{150n}$ is an upper bound for $\dfrac{c_n}{n}$. The fact that $\dfrac{8}{15} \approx .5\overline{3}$ concurs with our earlier finding that c_n is asymptotically 53.2% of n. The upper bound is appropriately slightly larger that this percentage. In fact, the integer part of $\dfrac{8n}{15} + \dfrac{43}{150}$ is almost always the same as that of c_n, making it a reasonably reliable approximation.

Conclusion

We have found a strategy for Player One. She must find the positive root c_n of the cubic: $2u^3 + (3+6n)u^2 + u - n(n+1)(2n+1) = 0$, and evaluate for her specific value of n. She should spin again if she spins a number lower than c_n on her first spin, but she should stay if she spins a number higher than or equal to c_n.

We have also found that the value of c_n will always be between $\frac{n}{2}$ and $\frac{8n}{15}+\frac{43}{150}$.

Further research will include investigating what Player One's strategy will be when more players are added. This will also expand to question the other players' strategies. It would also be interesting to look at what happens when the rules are changed; if the players are allowed more than two spins, or they have another spin-off on ties.

References

[1] Stewart, J. *Calculus: early transcendentals.* Pacific Grove, CA: Brooks/Cole Publishing Company (1999).

[2] Wackerly, D., Mendenhall III, W., & Scheaffer, R. *Mathematical statistics with applications.* Pacific Grove, CA: Wadsworth Group (2002).

[3] Wolfram Research, Inc. Mathematica, Version 4.2. Champaign, IL (2002).

Thanks

I would like to thank Dr. Janet Bowers of San Diego State University for posing this question. I would also like to thank my mentors Dr. Todd Lee and Dr. Crista Coles for their help with this research.

About the author:

I am a senior Math and Theatre Arts double major at Elon University in North Carolina.

Aja Johnson
5900 Campus Box
Elon, NC 27244
ajohnson3@elon.edu