ON THE THEORY OF SOLID KNOTS

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Translated by Ted Ashton, 2004

1. Problem Statement and Results

In the past, chemical research has provided many applications for Knot Theory. For example, knots and links have been found in the molecular structure of certain chemical compounds. A cycloparaffin[1] molecule with over 50 carbon atoms can exist, for example, as a simple ring or as a knotted ring resembling a trefoil[2] (see figure 1). H. L. Frisch and E. Wassermann, in their work[1], call these "topological isomers" and note that they differ in some, predominately physical, characteristics. G. Schill and C. Zürcher report in [5] on mechanically linked molecules — Catenanes and Rotaxanes. For example, the simplest Catenane is represented by two linked rings (see figure 2).

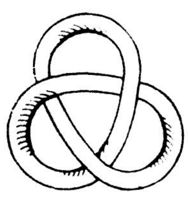

Figure 1

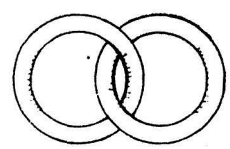

Figure 2

In classical knot theory, a *knot* is the image of S^1 in $\mathbb{R}^3$ and thus one-dimensional; *links* are combinations of knots and hence also one-dimensional. However the previously-mentioned chemical examples are of knotted and linked three-dimensional objects and classical Knot Theory is only partially applicable to these. So we would like to call these objects "solid knots".[3]

The study of the geometric characteristics of solid knots and links will involve, apart from topological questions (such as the question of knot type or isotopy class), also "metric questions."[4] So, for example, a

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[1]Also called "cycloalkane". Note that all footnotes are the translator's.

[2]German: *Kleeblattschlinge*, literally "cloverleaf loop".

[3]German: *massive Knoten*. As will soon be seen, Krötenheerdt and Veit include in this definition what are currently termed "thick knots" — the "solid knots of uniform thickness d" of section 4 — and also knots formed from congruent components, like pearl necklaces.

[4]German: *metrische Fragen*. One presumes these to be questions of distance and size and so forth. The sentence is perhaps better translated, "The study of the characteristics of solid knots and links involves not only topological questions but also geometric ones."

solid knot constructed of congruent components must have some minimum number of those components for its core[5] to form a given knot (for instance a trefoil or a particular rosette knot[6] R_n^m).

In what follows we will focus on two special kinds of solid knots, namely solid knots built from congruent components, in section 3, and solid knots of uniform thickness[7], in section 4. Section 2 will prepare the way for those via a lemma regarding certain polygonal knots (and here we use the term in the classical sense). We will restrict our attention to certain torus knots, those which in [2] and [3] are called "rosette knots of second order" R_2^m ($m = 3, 5, \dots$); the most basic representative of these second order rosette knots[8] is the trefoil, R_2^3. If we denote by $n_0(R_2^m)$ the minimum number of edges which are necessary to form a polygonal R_2^m knot, then the lemma will give us that $n_0(R_2^m)$ is bounded above by $m + 3$ and in the $m = 3$ case it is actually $n_0(R_2^3) = 6$.

We define as "Solid Knots made of Congruent Components" finite unions of congruent bodies B_i ($i = 1, \dots, n$) in $\mathbb{R}^3$ with the following characteristics:

(a) There exists a polygonal knot K — the core — which intersects the interior of each B_i.
(b) The B_i — the "components" — are convex and bounded.
(c) No two components share an interior point, but the two components B_i and B_{i+1} share at least one boundary point (defining B_{n+1} to be B_1).
(d) K intersects the boundary of each B_i in exactly two points, of which one is also in the boundary of B_{i-1} and the other in the boundary of B_{i+1} (defining B_0 as B_n).
(e) For each B_i, the part of K contained in B_i, along with the shortest curve on the boundary of B_i necessary to make a closed curve, forms a trivial knot.

For those solid knots we use the symbol $\mathbf{K}(K, B)$ where B is any representative of the B_i (which will be referred to as "components of the type B").

(Although the part of (c) which says that B_i and B_{i+1} share a point can be derived from (d), it was stated here explicitly for the sake of later emphasis on the geometry of the situation).

We denote by $n_0(K, B)$ the minimum number of congruent components of type B needed to form a solid knot $\mathbf{K}(K, B)$ for a given knot type K. The following can be shown:

1. If $K := R_2^m$ ($m = 3, 5, \dots$) and $B := \mathcal{C}$ (cube) then $n_0(R_2^m, \mathcal{C}) \leq 6m + 6$ (this estimate also applies for the Archimedean solids[9] $(3, 8, 8)$, $(4, 6, 6)$, $(4, 6, 8)$, $(3, 4, 3, 4)$, $(3, 4, 4, 4)$).
2. If $K := R_2^m$ ($m = 3, 5, \dots$) and $B := \mathcal{B}$ (ball) then $n_0(R_2^m, \mathcal{B}) \leq 4m + 4$.
3. If $K := R_2^m$ ($m = 3, 5, \dots$) and $B := \mathcal{K}_a$ (*Kalotte*[10] – a special portion of a ball which has significant applications and is better described in section 3) then $n_0(R_2^m, \mathcal{K}_a) \leq 13m + 9$.

[5]Here the authors use the term *Seele*, meaning "soul" or "spirit" and define it as a knot in the classical sense imagined to be inside the solid knot.

[6]German: *Rosettenknoten*. In [2], Krötenheerdt defines the term "*Rosettenknoten* R_n^m of order n" to mean closed braids of n strands of the form

$$Z = \begin{cases} (s_1 s_2^{-1} s_3 s_4 - 1 \cdots s_{n-2}^{-1} s_{n-1})^m & \text{for } n \text{ even}, \\ (s_1 s_2^{-1} s_3 s_4 - 1 \cdots s_{n-2} s_{n-1}^{-1})^m & \text{for } n \text{ odd}, \end{cases}$$

where $n > 1$ (naturally), $m > 1$ (or the knot is the unknot) and $(m, n) = 1$ (or the knot becomes a link).

[7]Now commonly called "thick knots".

[8]Specifically the $T(m, 2)$ torus knots.

[9]German: *halbregulären Polyeder*, literally "halfregular polyhedra". In the order listed these are: the truncated cube, the truncated octahedron, the truncated cuboctahedron (or great rhombicuboctahedron), the cuboctahedron, and the (small) rhombicuboctahedron. The symbols are vertex configurations, sometimes called "C&R symbols" and describe the faces which meet at any given vertex (cf. the parenthetical remark on page 5 regarding the corollary). The truncated octahedron, for example, has a square and two hexagons meeting at every vertex.

[10]The word *Kalotte* can be translated "calotte" (the skullcap worn by clergy in the Roman Catholic church), "crown," or "roof" among other things. As the shape is defined in this paper and does not appear to have given name in the mathematical realms, the translator has chosen to leave the German word but to treat it henceforth as an English noun with plural "kalottes".

We define as “Solid Knots of Uniform Thickness d” the unions of congruent balls, $\mathcal{B}_t$ $(0 \le t \le 1)$, in $\mathbb{R}^3$ with the following characteristics:

(a) There is a knot K – the core of the solid knot – on which the centers, C_t, of the $\mathcal{B}_t$ lie.
(b) The $\mathcal{B}_t$ have diameter d where d is the thickness of the solid knot.
(c) As t varies montonically along the interval 0 to 1, C_t varies montonically along the knot K and at $\mathcal{B}_1$ has come back around to C_0.
(d) Each ball $\mathcal{B}_t$ intersects the border of the union of the $\mathcal{B}_t$ exactly in the points of a great circle.

For these solid knots we use the symbol $\mathbf{K}(K, d)$ and occasionally shorten it to $\mathbf{K}$.

(In the above definition, it does not matter whether the $\mathcal{B}_t$ are understood to be solid balls or simply spherical surfaces. Due to characteristic (d), the core K cannot have corners and cannot penetrate the surface of $\mathbf{K}(K, d)$.)

We denote by $l_0(K, d)$ the minimum length of the core K of a thick knot $\mathbf{K}(K, d)$ given a particular knot type for K and a thickness d. It can be show that with $K := R_2^m$ we have

$$l_0(R_2^m, d) \le \left(\frac{1 + 7m}{2} + \frac{11 + 5m}{2} \cdot \frac{\pi}{4} \right) \cdot d \quad (m = 3, 5, \dots).$$

From this it follows, for example, that for the trefoil R_2^3

$$l_0(R_2^3, d) \le \left(11 + 13 \cdot \frac{\pi}{4} \right) \cdot d = d \cdot 21.210 \dots$$

In sections 3 and 4 some results of previous investigations into solid links are mentioned; further remarks about them are planned for a future paper. Further investigations could consider other types of components, find better bounds on $n_0(K, B)$ and $l_0(K, d)$ and, for particular knot types, accurate values for $n_0(K, B)$. Also, it seems possible and desirable to carry over results already known about rosette knots of order n and/or other knot classes.

2. A Lemma about Polygonal Knots

There exists a minimum number, $n_0(K)$, of edges needed to represent a given knot K as a polygonal knot in regular projection. In particular

Lemma 1. *Every polygonal knot of the type* $R_2^m (m = 3, 5, \dots)$ *can be formed with* $m + 3$ *edges; thus*

$$n_0(R_2^m) \le m + 3. \tag{1}$$

Proof. A trefoil R_2^3 is representable by six edges, as one can see from the standardized regular projection given in figure 3. (The points A_i and B_i in the following text and in figures 3 and 4 have indices $3, 5, \dots$ from the m of the rosette knot R_2^m). Now we can form an R_2^5 with eight edges by replacing the edge A_3B_3 by the chain $A_3A_5B_5B_3$, with A_3A_5 running under DE and B_5B_3 crossing over DE and the edge A_5B_5 (in the standardized regular projection of the knot R_2^5) lying parallel to DE outside the triangle DES, not crossing any other edge. So R_2^5 can be formed with $6 - 1 + 3 = 5 + 3$ edges while R_2^3 can be formed with $3 + 3$ edges.

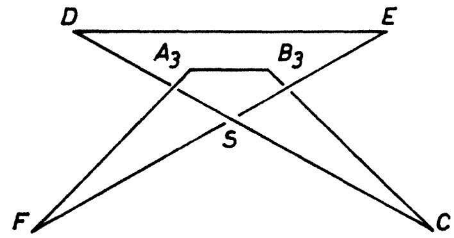

FIGURE 3

By induction it follows that we can form an R_2^{m+2} knot from

$$(m+3)-1+3=(m+2)+3$$

edges by proceeding from the standardized regular projection of an R_2^m (consisting of $m+3$ edges) and replacing the edge A_mB_m with the chain $A_mA_{m+2}B_{m+2}B_m$ in such a way that

(a) in the cases $m=3,7,\dots$, the edge A_mA_{m+2} crosses under the edge DE and $B_{m+2}B_m$ crosses over DE, while the edge $A_{m+2}B_{m+2}$ in the standard regular projection of the knot R_2^{m+2} lies parallel to DE, outside the triangle DES, and crosses no other edge of R_2^{m+2};

(b) in the cases $m=5,9,\dots$, the edge A_mA_{m+2} crosses over the edge DE and $B_{m+2}B_m$ crosses under DE, while the edge $A_{m+2}B_{m+2}$ in the standard regular projection of the knot R_2^{m+2} lies parallel to DE, inside the triangle DES, and crosses no other edge of R_2^{m+2} (see figure 4).

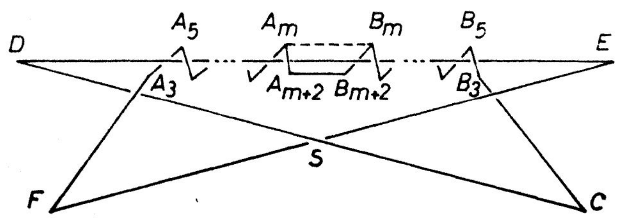

FIGURE 4

Thus we have shown that any polygonal knot of the type $R_2^m (m=3,5,\dots)$ can be formed with $m+3$ edges.

Remark 1. For the trefoil we have exactly

$$n_0(R_2^3)=6,$$

since any polygonal knot other than the unknot requires at least 6 edges (there are no knotted pentagons or squares).

Remark 2. For each $m=2,4,\dots$, R_2^m is a concatenation of two trivial knots. The estimate (1) applies also to $m=4,6,\dots$, as one can show by an appropriate construction procedure like the one in the proof of the above lemma; in this case $n_0(R_2^m)$ is understood to be the sum of the number of edges in the two trivial knots which form the R_2^m. In the case $m=2$ we have

$$n_0(R_2^2)=6$$

(both of the link components are chains of three edges each).

3. Solid Knots made of Congruent Components

In this section we will turn to the investigation of solid knots formed from congruent components. If we strengthen the requirement that two succesive components must share a point and instead insist that successive components with flat faces must share an entire face, then it seems that, e.g., from the regular polyhedra, cubes would be suitable components.

For solid knots made of congruent cubes we prove the following theorem.

Theorem 1. *Any solid knot made of congruent cubes $\mathcal{C}$ with core $K = R_2^m$ $(m = 3, 5, \dots)$ can be formed with $6m + 6$ cubes; so for $\mathbf{K}(R_2^m, \mathcal{C})$ we have*

(2) $$n_0(R_2^m, \mathcal{C}) \leq 6m + 6 \quad (m = 3, 5, \dots).$$

Proof. Any solid knot $\mathbf{K}(R_2^3, \mathcal{C})$ can be built from $24 = 6 \cdot 3 + 6$ cubes (see figure 5a; figure 5b shows the core with the centers of the cubes marked). The realization of the solid knot can be accomplished with $\mathbf{K}(R_2^5, \mathcal{C})$, $24 + 12 = 36 = 6 \cdot 5 + 6$ cubes (see figure 6; the centers of the additional cubes are shown by small circles).

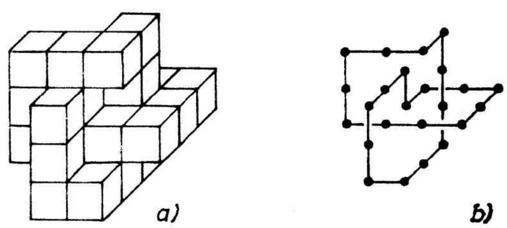

Figure 5

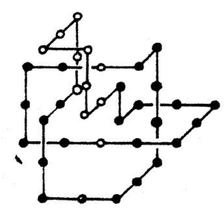

Figure 6

If one uses the same basic construction as for the polygonal knots (see section 2), then each solid knot $\mathbf{K}(R_2^{m+2}, \mathcal{C})$ can be created from the solid knot $\mathbf{K}(R_2^m, \mathcal{C})$ in the same way as the solid knot $\mathbf{K}(R_2^5, \mathcal{C})$ was created from $\mathbf{K}(R_2^3, \mathcal{C})$, that is, by inserting 12 additional cubes. Given that

$$n_0(R_2^m, \mathcal{C}) \leq 6m + 6,$$

it then follows that

$$n_0(R_2^{m+2}, \mathcal{C}) \leq 6m + 6 + 12 = 6(m + 2) + 6.$$

Hence by the Principle of Mathematical Induction we have the desired result.

Corollary 1. *The bound* (2) *remains valid if the cubes $\mathcal{C}$ are replaced with any of the Archimedean solids* $(3, 8, 8)$, $(4, 6, 6)$, $(4, 6, 8)$, $(3, 4, 3, 4)$ *or* $(3, 4, 4, 4)$.

(The symbol $(n_1, n_2, \dots, n_i)$ means that at every corner of the polyhedron, an n_1-gon, an n_2-gon, . . ., an n_i-gon meet in that order.) The corollary follows directly simply by noticing that the Archimedean solids mentioned all result from trimming cubes in various ways so every one of the cube faces contains a face of the Archimedean solid.

Remark. Theorem 1, as stated, applies only if the intersection of two neighbouring cubes is exactly one face. It is easy to see that the bound (2) applies even more if we drop the stronger requirement and only demand that neighboring components possess at least one common point.

If we use balls as components, we get

Theorem 2. *Any solid knot formed from congruent balls $\mathcal{B}$ around the core $K = R_2^m$ $(m = 3, 5, \dots)$ can be realised using $4m + 4$ balls; so for $\mathbf{K}(R_2^m, \mathcal{B})$ we have*

(3) $$n_0(R_2^m, \mathcal{C}) \leq 4m + 4 \quad (m = 3, 5, \dots).$$

Proof. Any solid knot $\mathbf{K}(R_2^3, \mathcal{B})$ can be formed with $16 = 4 \cdot 3 + 4$ balls (see figure 7). As explanation, imagine the centers of eight of the balls lying in one plane p; the centers of these balls are A, F, G_3, H_3, I_3, J, K and N_3 (the index on certain of the points coming from the m in R_2^m).

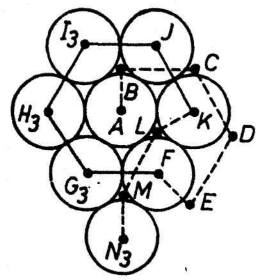

FIGURE 7

The centers of six additional balls lie in a plane parallel to p which is "under" p in figure 7; the centers of these balls are B, C, D and E and also L and M, and the distance between the two planes is fixed so that A, I_3, J and B form a regular tetrahedron. With two more appropriately-placed balls with centers O_3 and P_3 "over" p, we can make a connection between the ball centered at N_3 and the one centered at A. The closed path $ABCDEFG_3H_3I_3JKLMN_3O_3P_3A$ is a polygonal knot of type R_2^3 and with the associated balls forms the solid knot $\mathbf{K}(R_2^3, \mathcal{B})$.

A solid $\mathbf{K}(R_2^5, \mathcal{B})$ can be formed using $16 + 8 = 24 = 4 \cdot 5 + 4$ balls. In order to see this, we supplement the existing balls of the $\mathbf{K}(R_2^3, \mathcal{B})$ with four new balls in the plane p whose centers are the points G_5, H_5, I_5 and T_5 (see figure 8). The path $G_3H_3I_3$ is replaced by the path $G_3G_5H_5I_5I_3$, and the point H_3 is renamed Q_5. Then the points O_3 and P_3 are shifted "over" p into suitable points O_5 and P_5 so that the balls around these points form a connection between the ball centered at N_3 and the one centered at Q_5.

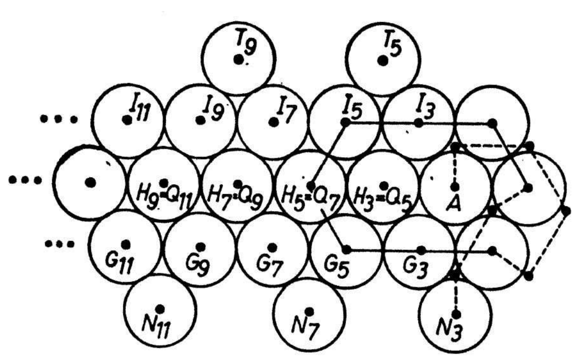

FIGURE 8

With two additional balls around suitable points R_5 and S_5 "under" p we can connect the ball centered at Q_5 with the one at T_5; and finally, two further new balls around appropriate points U_5 and V_5 "over" p can form a connection between the ball around T_5 and the one at A.

If one uses the previously-discussed method of constructing the R_2^m $(m = 3, 5, \dots)$ polygonal knots, then every solid knot $\mathbf{K}(R_2^{m+2}, \mathcal{B})$ can be built from a $\mathbf{K}(R_2^m, \mathcal{B})$ in the same fashion as the $\mathbf{K}(R_2^5, \mathcal{B})$ was created from a $\mathbf{K}(R_2^3, \mathcal{B})$ by carefully adding eight balls:

(a) In the cases $m = 3, 7, 11, \ldots$ we supplement the original balls of the $\mathbf{K}(R_2^m, \mathcal{B})$ with four new balls, whose centers, G_{m+2}, H_{m+2}, I_{m+2} and T_{m+2} lie in the plane p (see figure 8). Then we replace the path $G_m H_m I_m$ with the path $G_m G_{m+2} H_{m+2} I_{m+2} I_m$ and rename the point H_m to Q_{m+2}. Next we shift the points O_m and P_m "over" p into suitable positions O_{m+2} and P_{m+2} so that the balls around these points form a connection between the ball around N_m and the one around Q_{m+2}. Then we can add two more balls around selected points R_{m+2} and S_{m+2} to form a connection "under" p between the ball around Q_{m+2} and the one around T_{m+2} and similarly we can add two balls "over" p to make a connection between the ball at T_{m+2} and the one around Q_m (where $Q_3 := A$).

(b) In the cases $m = 5, 9, 13, \ldots$ we supplement the original balls of the $\mathbf{K}(R_2^m, \mathcal{B})$ by four new balls with their centers, G_{m+2}, H_{m+2}, I_{m+2} and N_{m+2}, lying in the plane p (see figure 8). So the path $G_m H_m I_m$ gets replaced by the path $G_m G_{m+2} H_{m+2} I_{m+2} I_m$ and we rename H_m to Q_{m+2}. Next we move the points R_m and S_m "under" p over to suitable points R_{m+2} and S_{m+2} so that the balls around these points form a connection between the ball around Q_{m+2} and the ball around T_m. Two more balls centered at the appropriate points Q_{m+2} and P_{m+2} "over" p suffice to form a connection between the balls around N_{m+2} and Q_{m+2}; a final two balls at L_{m+2} and M_{m+2} "under" p will connect the ball around Qm with the one around N_{m+2}.

So given

$$n_0(R_2^m, \mathcal{B}) \leq 4 \cdot m + 4 \quad (m = 3, 5, \ldots)$$

it follows that

$$n_0(R_2^{m+2}, \mathcal{B}) \leq 4 \cdot m + 4 + 8 = 4(m + 2) + 4.$$

Hence by the Principle of Mathematical Induction we have Theorem 2.

Of particular importance — particularly regarding applications in Chemistry — is the investigation of solid knots formed from special congruent ball pieces — the so-called "kalottes". These are formed by taking a ball, $\mathcal{B}_1$ of radius r_1 and slicing it with two planes, p_1 and p_2 which are at an angle of $\frac{\pi}{3}$ to each other and are both tangent planes of a ball, $\mathcal{B}_2$, which is concentric with $\mathcal{B}_1$ and has radius $r_2 = \frac{\sqrt{3}}{3} r_1$. These shapes model the situation when the diameters of the normal atomic spheres are larger than the distances between centers of neighboring atoms in certain bonds so that neighboring atomic spheres intersect. As the original concept of the kalotte is one of intersecting balls, we will assume in what follws that two neighboring kalottes intersect in a disk. Then for solid knots formed from these kalottes, $\mathcal{K}_a$, we have

Theorem 3. *Any solid knot built from congruent kalottes around a core $K = R_2^m$ $(m = 3, 5, \ldots)$ can be formed with $13m + 9$ kalottes; thus we have the bound*

$$n_0(R_2^m, \mathcal{K}_a) \leq 13m + 9 \quad (m = 3, 5, \ldots). \tag{4}$$

Proof. The construction of solid knots from the above congruent kalottes can be based on the construction of solid knots formed from congruent balls. If the core of a solid knot makes an angle of $\frac{2\pi}{3}$ within a ball of radius r centered at M, we can replace the ball with three kalottes (see figure 9).

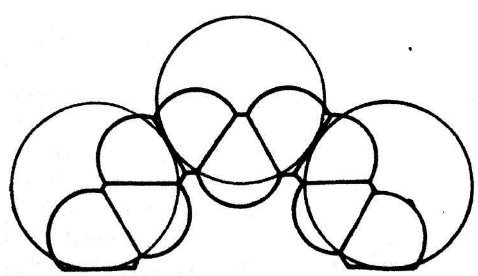

FIGURE 9

But we cannot simply multiply the number of balls of the solid knot $\mathbf{K}(R_2^3, \mathcal{B})$ of figure 7 by 3 to get an upper bound for $n_0(R_2^3, \mathcal{K}_a)$ since the core R_2^3 doesn't form an angle of $\frac{2\pi}{3}$ at B, E, K and at least one of N_3, P_3, O_3 or A. So we select for the polygonal knot of type R_2^3 the new arrangement given in figure 10.

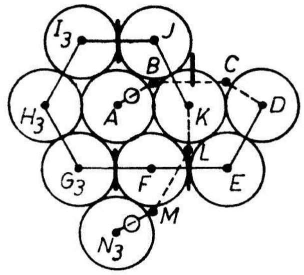

FIGURE 10

We place three kalottes in each of the balls around J, K, L, M and B in figure 9; we can do this because the angles at corners J, K, L, M and B of the polygonal knot each have an angle of $\frac{2\pi}{3}$. Four kalottes, in the arrangement shown in figure 11a, replace the ball around F. Since the knot makes an angle of $\frac{\pi}{2}$ at D, we remove the balls around C, D and E and replace them by a "bridge" formed from nine kalottes as in figure 11b; that is possible because the point B' (the former point of contact between around B and the ball around C) and the point E' (the former point of contact between the ball around E and the one around F) are separated by a distance $|B'E'| = r \cdot 2 \cdot \sqrt{2} = r \cdot 2.8\ldots$ and the "bridge" can adjust to any length from $2 \cdot \sqrt{3} \cdot r$ through $\frac{2}{3} \cdot \sqrt{3} \cdot r$ to meet up with B'' and E'' by suitable turns of kalottes 1 and 2 around the axis a_2 and kalottes 8 and 9 around axis a_1.

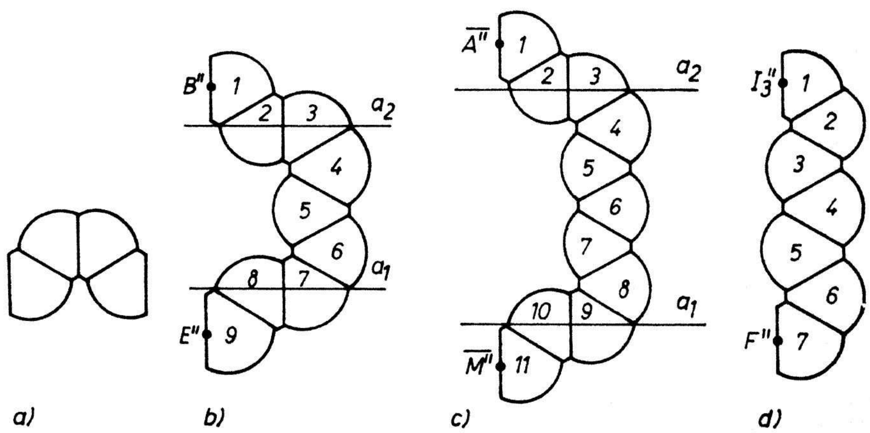

FIGURE 11

Then at A' (the point of contact betweeen the ball around A and the one around B) and at M' (the point of contact between the balls at M and N_3) the respective kalottes have a circular face lying parallel to the plane p; furthermore, they lie in the same plane and the distance between the centers of those faces — which we will term $\bar{A}'$ and $\bar{M}'$ respectively — is $|\bar{A}'\bar{M}'| = |AN_3| = 2 \cdot r \cdot \sqrt{3} = r \cdot 3.4\ldots$ so we connect the kalotte at A' and the kalotte at M' with a "bridge" built from eleven kalottes as in figure 11c; that is possible because in this "bridge" the distance between $\bar{A}''$ and $\bar{M}''$ can be set to any value between $r \cdot 4 \cdot \frac{2}{3} \cdot \sqrt{3}$ and $r \cdot 2 \cdot \frac{2}{3} \cdot \sqrt{3}$ by suitable turns of kalottes 1 and 2 around axis a_2 and kalottes 10 and 11 around axis a_1.

Although the knot forms an angle of $\frac{2\pi}{3}$ at each of the corners G_3, H_3 and I_3 and we could place the usual three kalottes in each place, we instead replace those three balls with a "bridge" built of seven kalottes as in figure 11d. That is possible because the points F'' and I_3'' of the "bridge" naturally have a distance $r \cdot 2 \cdot \sqrt{3}$ from the seven kalottes and because the point F' (the former point of contact between the ball around F and the ball around G) and the point I_3' (which was were the ball around I_3 met up with the ball around J) were also $r \cdot 2 \cdot \sqrt{3}$ apart.

With the procedure described we have arranged

$$5 \cdot 3 + 4 + 9 + (2 + 11) + 7 = 48 = 13 \cdot 3 + 9$$

congruent kalottes in such a way that every kalotte shares a circular face with each of two other kalottes. If one imagines the center of each kalotte connected by an edge with the center of the two neighbor kalottes which share a face with it, then one has a closed path of 48 edges forming a polygonal R_2^3 knot. This forms the solid knot $\mathbf{K}(R_2^3, \mathcal{K}_a)$ and so we have $n_0(R_2^3, \mathcal{K}_a) \leq 13 \cdot 3 + 9$.

In forming a $\mathbf{K}(R_2^5, \mathcal{K}_a)$ from a $\mathbf{K}(R_2^3, \mathcal{K}_a)$, we first (as instructed in the proof of Theorem 2) add four balls to the $\mathbf{K}(R_2^3, \mathcal{B})$ shown in figure 10 around the points G_5, H_5, I_5 and T_5. Then we replace the balls around G_3 and I_3 each with the four kalottes of figure 11a. The balls around G_5, H_5 and I_5 are again replaced by the "bridge" of seven kalotten found in figure 11d. The balls around O_5, P_5, U_5 and V_5 can be placed in such a way that the polygonal R_2^5 has an angle of $\frac{2\pi}{3}$ at N_3, O_5, P_5, U_5 and A so each of these six balls can be replaced with three kalottes as desired. For the connection between the ball around Q_5 and the ball around T_5 we use the same "bridge construction" underneath the plane p as with the connection between A and N_3 above p in building $\mathbf{K}(R_2^3, \mathcal{K}_a)$, i.e., we need two kalottes and also a "bridge" of eleven kalottes. The remaining balls of the $\mathbf{K}(R_2^5, \mathcal{B})$ are replace with kalottes in the same way as during the building of the solid knot $\mathbf{K}(R_2^3, \mathcal{K}_a)$. So we added $2 \cdot 4 + 6 \cdot 3$ kalottes to the 48 kalottes of the solid knot $\mathbf{K}(R_2^3, \mathcal{K}_a)$ in order to get the solid knot $\mathbf{K}(R_2^5, \mathcal{K}_a)$. Since $48 + 2 \cdot 4 + 6 \cdot 3 = 48 + 2 \cdot 13$ we have $n_0(R_2^5, \mathcal{K}_a) \leq 13 \cdot 5 + 9$.

The construction given in the proof of Theorem 2 lets a $\mathbf{K}(R_2^{m+2}, \mathcal{B})$ be built from a $\mathbf{K}(R_2^m, \mathcal{B})$ (for $m = 3, 5, \dots$) by the addition of 8 balls, but for each m those eight balls are situated as above, causing $2 \cdot 4 + 6 \cdot 3 = 2 \cdot 13$ kalottes to be added in the transition from a $\mathbf{K}(R_2^m, \mathcal{K}_a)$ to a $\mathbf{K}(R_2^{m+2}, \mathcal{K}_a)$, i.e.,

$$n_0(R_2^m, \mathcal{K}_a) \leq 13 \cdot m + 9$$

implies that

$$n_0(R_2^{m+2}, \mathcal{K}_a) \leq 13 \cdot (m + 2) + 9$$

for each $m = 3, 5, \dots$ and so Theorem 3 is established.

For a "solid trefoil" from congruent kalottes — the name we want to use for $\mathbf{K}(R_2^3, \mathcal{K}_a)$ — it follows from Theorem 3 that 48 kalottes are sufficient. (In [1], H. L. Frisch and E. Wassermann determine that a Cycloparaffin must consist of at least 50 carbon atoms to form a "trefoil". Here "trefoil" means the same as our $\mathbf{K}(R_2^3, \mathcal{K}_a)$. The 50 carbon atoms mentioned are understood to partially intersect, so the kalotte shape described is appropriate to the situation.)

The considerations and construction procedures for solid knots made of congruent components which we have so far can be tranferred over to links formed from congruent-component solid knots. For cubes, for balls and for the described ball pieces (kalottes), some such links have been studied — ones where the "rosette links of second order," R_2^m $(m = 2, 4, \dots)$, form the core; the simplest representative of the "Rosette links of second order" is R_2^2 — two linked rings. Regarding links formed from congruent-component solid knots we can also ask the question about the minimum number of components necessary for a given "link core" of type V and congruent components of kind B. We designate this minimum number $n_0(V, B)$; it is worth noting that each link consists of several knots which do not all have to contain the same number of components.

We include here, without proof, the following bounds for $V := R_2^m$ $(m = 2, 4, \dots)$, $B := \mathcal{C}$ (cubes), $B := \mathcal{B}$ (balls) and $B := \mathcal{K}_a$ (kalottes — the shape described above):

$$n_0(R_2^m, \mathcal{C}) \leq 6 \cdot m + 4 \quad (m = 2, 4, \dots), \tag{5}$$

$$n_0(R_2^m, \mathcal{B}) \leq 4 \cdot m + 2 \quad (m = 4, 6, \dots), \tag{6}$$

$$n_0(R_2^m, \mathcal{K}_a) \leq 13 \cdot m + 4 \quad (m = 4, 6, \dots). \tag{7}$$

(Each of the links of congruent-component solid knots above consists of two pieces; during the constructions a particularly simple representation was chosen for each piece.)

Remark 1. In the case $m = 2$, instead of the bound (6) we have the actual value

$$n_0(R_2^2, \mathcal{B}) = 12.$$

Supposing that we have five balls arranged so that each ball touches two others and their centers are at the corners of a regular pentagon, is it possible to place a sixth ball so that it touches, inside the pentagon, the plane containing the centers of the other balls? Well, the height h of a pyramid whose base is a regular pentagon and whose sides are equilateral triangles with sidelengths a is

$$h = \frac{a}{2}\sqrt{2\left(1 - \frac{\sqrt{5}}{5}\right)} > \frac{a}{2}.$$

Remark 2. It is possible to build a trival solid knot (i.e. a solid knot whose core is equivalent to a circle) with 18 kalottes (see figure 9) and to link this with a second similar knot. In the work mentioned [1], it turns out the simplest catenane in the form of two linked rings (see figure 2) needs at least 20 carbon atoms per ring. It is also mentioned that it is possible for two 18-unit carbon rings to link with[11] each other but that due to the inter-atomic repulsions two more atoms must be inserted in each ring. Furthermore it turns out in [1] that at least 66 $(= 20 + 26 + 20)$ carbons atoms are required for a chain of three carbon rings. If one imagines such a chain from three trivial congruent-kalotte knots, it is easy to see that the minimum number of kalottes necessary is not more than 62 $(= 18 + 26 + 18)$.

4. Solid Knots of Uniform Thickness d

We now turn our attention to solid knots of uniform thickness d. We get an estimate for the value of $l_0(R_2^m, d)$ from examining solid knots formed from congruent cubes. It is possible to embed into any solid knot $\mathbf{K}(R_2^m, \mathcal{C})$ which was designed according to the plan in the proof of Theorem 1 a solid knot of uniform thickness d, where d is the edge length of the cubes. The R_2^m core of the $\mathbf{K}(R_2^m, \mathcal{C})$ we designed consists of $6m+6$ edges each of length d. This core, in the case $m = 3$, forms a right angle at 13 of its vertices and and a straight angle at the other 11. The method used to produce a $\mathbf{K}(R_2^{m+2}, \mathcal{C})$ from a $\mathbf{K}(R_2^m, W)$ $(m = 3, 5, \dots)$ involved adding twelve cubes. The core R_2^{m+2} which arose from the core R_2^m in this way had twelve new edges which met at five of the new vertices in right angles and at seven of them in straight angles. So the core of a $\mathbf{K}(R_2^m, \mathcal{C})$ $(m = 3, 5, \dots)$ makes

$$\text{a right angle at } 13 + \frac{m-3}{2} \cdot 5 \text{ vertices}$$

and

$$\text{a straight angle at } 11 + \frac{m-3}{2} \cdot 7 \text{ vertices.}$$

In each cube containing a vertex at which the core knot is straight, we simply keep the core as it is. In the cubes where the core makes a right angle, we replace the portion of the core in the cube with a circular

[11] German: *verschlingen*, literally “devour”.

elbow of radius $\frac{d}{2}$ in such a fashion that the resulting closed curve forms the core of a $\mathbf{K}(R_2^m, d)$ embedded in the $\mathbf{K}(R_2^m, \mathcal{C})$. For a solid knot of uniform thickness d thus constructed, we have now found that

$$l_0(R_2^m, d) \leq \left(11 + \frac{m-3}{2} \cdot 7\right) \cdot d + \left(13 + \frac{m-3}{2} \cdot 5\right) \cdot \frac{\pi}{4} \cdot d \qquad (m = 3, 5, \dots).$$

So then we immediately have the following theorem:

Theorem 4. *For any given solid knot of uniform thickness d with core knot R_2^m $(m = 3, 5, \dots)$,*

$$l_0(R_2^m, d) \leq \left(\frac{1+7m}{2} + \frac{11+5m}{2} \cdot \frac{\pi}{4}\right) \cdot d. \tag{8}$$

Hence in the $m = 3$ case we get a bound for the "solid trefoil" of uniform thickness d of

$$l_0(R_2^3, d) \leq \left(11 + 13 \cdot \frac{\pi}{4}\right) \cdot d,$$

i.e.

$$l_0(R_2^3, d) \leq d \cdot 21.210\dots$$

We wish to examine in the following, as well, the solid knots of greatest possible uniform thickness d which can be embedded in a $\mathbf{K}(R_2^m, \mathcal{K}_a)$ developed according to the construction given in the proof to Theorem 3. Every such $\mathbf{K}(R_2^m, \mathcal{K}_a)$ consists of $13m + 9$ kalottes and the core R_2^m makes an angle of $\frac{2\pi}{3}$ in the center of each kalotte. In each kalotte we replace the portion of the core within it with a $\frac{\pi}{3}$-radian circular arc[12] with radius $\sqrt{3} \cdot r_2$ (the *kalotte* under consideration, according to the definitions in section 3 is a ball $\mathcal{B}_1$ with center C_1 of radius r_1 truncated by two planes, p_1 and p_2, which meet at an angle of $\frac{\pi}{3}$ and are tangential to a ball $\mathcal{B}_2$ around C_1 of radius $r_2 := \frac{1}{3} \cdot \sqrt{3} \cdot r_1$); the radius $\sqrt{3} \cdot r_2$ was selected so that it is equal to the distance from the intersection of planes p_1 and p_2 to the centers of the circular faces. If the arcs are inserted into each kalotte in such a way that they meet the kalotte's circular faces perpendicularly in their centers, then the $13m+9$ arcs form a closed curve with knot type R_2^m. Since the circular faces of each kalotte have radius $\rho = \sqrt{{r_1}^2 - {r_2}^2} = \frac{1}{3} \cdot \sqrt{6} \cdot r_1$, we have thus found a solid $\mathbf{K}(R_2^m, d)$ knot of uniform thickness $d = \frac{2}{3} \cdot \sqrt{6} \cdot r_1$, which is embedded in $\mathbf{K}(R_2^m, \mathcal{K}_a)$; with the given embedding, a larger thickness d is not possible. The length of l of the core knot of the $\mathbf{K}(R_2^m, d)$ built in this fashion obeys

$$\begin{aligned} l &= (13m+9) \cdot \frac{2\pi}{6} \cdot \sqrt{3} \cdot r_2 = (13m+9) \cdot \frac{\pi}{3} \cdot \sqrt{3} \cdot \frac{1}{3}\sqrt{3} \cdot r_1 \\ &= (13m+9) \cdot \frac{\pi}{3} \cdot r_1 = (13m+9) \cdot \frac{\pi}{3} \cdot \frac{1}{4}\sqrt{6} \cdot d. \end{aligned}$$

This is a worse bound for $l_0(R_2^m, d)$ than the one given in Theorem 4. In the case $m = 3$ — the "solid trefoil" of uniform thickness d — this gives a length for the core of

$$l = 4 \cdot \pi \cdot \sqrt{6} \cdot d = d \cdot 30.786\dots\text{[sic]}$$

The results found were appplied also to links composed of knots with uniform thickness d whose core are the rosette links of second order R_2^m $(m = 2, 4, \dots)$. We state without proof that from the embedding of a link of solid knots $\mathbf{K}(R_2^m, d)$ of uniform thickness d into a link of solid knots $\mathbf{K}(R_2^m, \mathcal{C})$ of congruent cubes, one can get the bound

$$l_0(R_2^m, d) \leq \left(\frac{7m}{2} + \frac{8+5m}{2} \cdot \frac{\pi}{4}\right) \cdot d \qquad (m = 4, 6, \dots) \tag{9}$$

In the case $m = 2$ simple considerations lead directly to

$$l_0(R_2^2, d) = 2 \cdot \pi \cdot d.$$

[12] German *Sechstelkreisbogen*. *Sechstel* means "one-sixth" and *Kreisbogen* is a circular arc.

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