Notes to Problems from Section 3.6

3.6.2. (Parts a), c), and e) were assigned.) Decide which sets of vectors are linearly independent.

a) *The set of two-by two matrices consisting of* $A := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, C := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
These are independent. For the proof, start out by assuming that
$c_1A + c_2B + c_3C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and equate entries to show $c_1 = c_2 = c_3 = 0$.

b) *The set of functions in* $\mathcal{P}_1$ *whose members are* $f_1(t) = t$, $f_2(t) = t + 1$, *and* $f_3(t) = t + 2$.
These are dependent because $f_1 + f_3 = 2f_2$.

c) *The set of functions in* $\mathcal{C}^\infty(\mathbb{R})$ *whose members are* $f_1(t) = 1$, $f_2(t) = \cos t$, *and* $f_3(t) = \sin t$.
These are independent. For the proof, assume $c_1f + c_2f_2 + c_3f_3 = 0$ (the zero function). This means $c_1 + c_2 \cos t + c_3 \sin t = 0$ for every real number t. Substitute $t = 0, \pi/2, \pi$ in this identity to get $c_1 + c_2 = c_1 + c_3 = c_1 - c_2 = 0$. The only solution to the latter equations is $c_1 = c_2 = c_3 = 0$ as desired.

d) *The set of functions in* $\mathcal{C}^0(\mathbb{R})$ *whose members are* $f_1(t) = 1$, $f_2(t) = \sin^2 t$, *and* $f_3(t) = \cos^2 t$.
These are dependent by the trig identity $f_2 + f_3 = f_1$.

e) *The set of functions in* $\mathcal{C}^\infty(\mathbb{R})$ *whose members are* $f_1(t) = 1$, $f_2(t) = \cos t$, *and* $f_3(t) = \cos 2t$.
The proof of Part c) shows these are independent.

f) *The set of functions in* $\mathcal{C}^\infty(\mathbb{R})$ *having members* $f_1(t) = 1$, $f_2(t) = \cos 2t$, *and* $f_3(t) = \cos^2 t$.
These are dependent by the trig identity $\cos 2t = 2\cos^2 t - 1$.

3.6.3. (Parts a), c), and e) were assigned.) Determine which of the following is a subspace of the space $\mathcal{M}_{2\times2}$ of two by two matrices. Give bases for those which are.

a) *The set* $\mathcal{S}$ *of matrices* A *having the vector* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *in their null spaces.*

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A belongs to $\mathcal{S}$ if and only if $a + 2b = c + 2d = 0$. Thus $\mathcal{S}$ is a 2-dimensional subspace of $\mathcal{M}_{2\times2}$. For a basis, you can take the two matrices $\begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix}$

b) *The set* $\mathcal{S}$ *of matrices* A *having the vector* $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *in their column spaces.*
This is not a subpace because it doesn't include the zero matrix.

c) *The set of matrices having rank one.*
This is not a subpace because it doesn't include the zero matrix.

d) *The set of matrices having rank* ≤ 1.
This does include the zero matrix, but it is not a subspace because it is not closed under addition.

e) *The set of matrices which are in echelon form.*
This also fails to be a subspace because it is not closed under addition.

f) *The set* $\mathcal{S}$ *of matrices* A *satisfying* $A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A$

Write $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A belongs to $\mathcal{S}$ if and only if $a = d$ and $b = c = 0$. Thus $\mathcal{S}$ is a one-dimensional subspace of $\mathcal{M}_{2\times 2}$. For a basis, you can take the identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

g) *The set* $\mathcal{S}$ *of matrices* A *satisfying* $A \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} A^T$

The answer is the same as for Part f).

3.6.6. (Parts b), d), e), and h) were assigned.) Determine which of the following are subspaces of $\mathcal{C}^0(\mathbb{R})$. Provide bases for those which are.

a) *The set* $\mathcal{S}$ *of continuous functions* f *satisfying* $f(1) = 2$.

This is not a subspace because it doesn't include the 0 function.

b) $\mathcal{S} := \{f \in \mathcal{P}_2 : \int_0^1 f(t)\,dt = 0\}$

This is a subspace because integration is linear, i.e. $\int_0^1 f + cg = \int_0^1 f + c\int_0^1 g$ for any integrable functions f, g and scalar c.

The most general member of $\mathcal{P}_2$ takes the form $f(t) = at^2 + bt + c$ for scalars a, b, c. Integrating we see that f belongs to $\mathcal{S}$ if and only if $\frac{a}{3} + \frac{b}{2} + c = 0$. Thus $\mathcal{S}$ is two-dimensional. One basis is given by the two polynomials $p(t) = 3t^2 - 1$ and $q(t) = 2t - 1$.

c) $\mathcal{S} := \{f \in \mathcal{C}^1(\mathbb{R}) : f'(t) + 2f(t) = 0 \text{ for all } t \in \mathbb{R}\}$

This is a subspace because differentiationation is linear. One member of $\mathcal{S}$ is given by $f(t) = \exp(-2t)$. Methods of calculus show that this single function is a basis for $\mathcal{S}$.

d) $\mathcal{S} := \{f \in \mathcal{P}_4 : f(t) - tf'(t) = 0 \text{ for all } t \in \mathbb{R}\}$

This is a subspace because differentiationation is linear. The most general member of $\mathcal{P}_4$ takes the form $f(t) = at^4 + bt^3 + ct^2 + dt + e$. This means $tf'(t) = 4at^4 + 3bt^3 + 2ct^2 + dt$. In order for the latter functions to be equal, we must have $a = b = c = e = 0$, but d can be arbitrary. Thus $\mathcal{S}$ has the single function $f(t) = t$ as a basis.

e) $\mathcal{S} := \{f \in \mathcal{P}_4 : f(t) - tf'(t) = 1 \text{ for all } t \in \mathbb{R}\}$

This is not a subspace because it does not include the 0 polynomial.

f) $\mathcal{S}_1 := \{f \in \mathcal{C}_2(\mathbb{R}) : f''(t) + f(t) = 0 \text{ for all } t \in \mathbb{R}\}$

g) $\mathcal{S}_2 := \{f \in \mathcal{C}_2(\mathbb{R}) : f''(t) - f'(t) + 6f(t) = 0 \text{ for all } t \in \mathbb{R}\}$

These are both subspaces because differentiation is linear. The theory of differential equations tells us that each is two-dimensional. The functions $\sin t$ and $\cos t$ form a basis for $\mathcal{S}_1$, while the two exponential functions $\exp(3t)$ and $\exp(-2t)$ form a basis for $\mathcal{S}_2$.

h) $\mathcal{S} := \{f \in \mathcal{C}_1(\mathbb{R}) : f(t) = \int_0^t f(s)\,ds = 0 \text{ for all } s \in \mathbb{R}\}$

Suppose $f \in \mathcal{S}$. Substituting $t = 0$ in the equation shows $f(0) = 0$. On the other hand differentiating the equation yields $f'(t) = f(t)$ for all t. Thus the only member of $\mathcal{S}$ is the 0 function.

3.6.7. *The set of solutions of a linear homogeneous differential equation is a subspace of* $\mathcal{C}^n(\mathbb{R})$

There is a lot more notation in the precise statement of this given in the text, but the proof amounts to noting that differentiation and multiplication by fixed continuous functionss respect addition and scalar multiplication in $\mathcal{C}^1(\mathbb{R})$.

Notes to Problems from Section 4.3

4.3.1a. *Calculate the standard matrix for the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by rotation (counterclockwise) through $-\pi/2$ radians about the origin followed by reflection across the line $x_1 - x_2 = 0$.*

Write R for the reflection and S for the rotation. Recall that the columns of the standart matrix of a linear transformation are the vectors to which it sends the basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Since R interchanges these vectors, we have $[R]_{std} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ Similarly $[S]_{std} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$. Thus $[T]_{std} = [R]_{std}[S]_{std} = \begin{bmatrix} -\sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}$.

4.3.5. Consider the basis $\mathcal{B}$ for $\mathbb{R}^2$ consisting of the vectors $v_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

a) *Given $[T]_{std} = \begin{bmatrix} 1 & 5 \\ 2 & -2 \end{bmatrix}$, find $[T]_{\mathcal{B}}$.*
Take P to be the matrix whose columns are v_1 and v_2. The change of basis formula gives us $[T]_{\mathcal{B}} = P^{-1}[T]_{std}P$, so all we have to do is substitute.

b) *Given $S(v_1) = 2v_1 + v_2$ and $S(v_2) = -v_1 + 3v_2$. find the standard matrix of S.*
The given information tells us $[S]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$, so all we have to do is substitute in the change of basis formula $[S]_{std} = P[S]_{\mathcal{B}}P^{-1}$.

4.3.9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ by reflecting across the plane $x_1 + x_2 + x_3 = 0$.

a) *Find an orthogonal basis $\mathcal{B} = \{v_1, v_2, v_3\}$ so that v_1, v_2 span the given plane and v_3 is orthogonal to it.*
Write V for the plane. Almost by definition, we at least know the vector $v_3 := (1, 1, 1)$ spans $V^\perp$. One orthogonal basis for V is given by $v_1 := (1, -1, 0), v_2 := (1, 1, -2)$, though there are of course many others. (Also, it is not really necessary for $v_1 \perp v_2$ in the rest of the problem.)

b) *Find the matrix of T relative to the basis $\mathcal{B}$.*
By contruction $Tv_1 = v_1, Tv_2 = v_2$, and $Tv_3 = -v_3$ so $[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

c) *Find the standard matrix of T.*
Take P to be the matrix having v_1, v_2, v_3 as its columns. By the change of basis formula, we have $[T]_{std} = P[T]_{\mathcal{B}}P^{-1}$.

4.3.20. Prove or give a counterexample.

a) *If B is similar to A, then B^2 is similar to A^2.*
This is true. Squaring both sides of the equation $B = P^{-1}AP$ yields $B^2 = P^{-1}A^2P$

b) *If B^2 is similar to A^2, then B is similar to A.*
This is false. For a counterexample, take A to be an identity matrix and $B = -A$.

c) *If B is similar to A, and A is non-singular, then B is non-singular.*
This is true. Taking determinants of both sides of the equation $B = P^{-1}AP$ yields $\det B = \det A$.

d) *If B is similar to A, and A is symmetric, then B is symmetric.*

This is false. For a counterexample, take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$.

e) *If B is similar to A, then $N(B) = N(A)$.*
This is false. The counterexample for Part d) works here as well.

f) *If B is similar to A, then* $\operatorname{rank}(B) = \operatorname{rank}(A)$.
This is true. When X, Y are matrices of the same size with Y invertible, then X and XY have the same column space and hence the same rank. Taking transposes shows X and YX also have the same rank. Thus $\operatorname{rank}(B) = \operatorname{rank}(PAP^{-1}) = \operatorname{rank}(PA) = \operatorname{rank}(A)$, as desired.

Notes to Problems from Earlier Sections of Chapter 3

3.1.1. The sets a),c),d),f), and i) are not subspaces because they do not include the zero vector (actually the space of Part f) is empty).

The set of Part b) satisfies all three conditions of the definition and hence is a subspace of $\mathbb{R}^3$.

The set of Part e) is a subspace because its lone member is the zero vector; it is not easy to verify Condition 2 of the definition directly.

The set of Part g) is a subspace by Proposition 1.2. The same is true for Part h) because $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$; note the contrast with part i).

3.1.14a. Let A be an m by n matrix and suppose W is a subspace of $\mathbb{R}^m$. Prove that $X := \{x \in \mathbb{R}^n : Ax \in W\}$ is a subspace of $\mathbb{R}^n$.

proof. Since $A0 = 0$, we see that $0 \in X$.

To see that X is closed under addition, suppose $x, y \in X$. By definition, this means Ax and Ay both belong to W. Because, W is a subspace, this implies that $A(x + y) = Ax + Ay \in W$. Thus $x + y \in X$ by definition, and we have shown that X is closed under addition. The proof that X is closed under scalar multiplication is similar.

3.2.12. Suppose A and B are m by n matrices. Then $C(A)$ and $C(B)$ are orthogonal subspaces iff $A^T B = 0$.

proof. Suppose first that $C(A) \perp C(B)$ and let $x, y \in \mathbb{R}^n$. Then $x \cdot A^T By = Ax \cdot By = 0$. Since this is true for every x, we conclude that $A^T By = 0$. Since this is true for every y, we conclude that $A^T B = 0$. and we have established the forward implication; reverse the argument for the backward implication.

3.3.1. The given vectors are dependent because $2v_1 + 0v_2 + (-1)v_3 = 0$. This can be discovered by inspection or by matrix reduction. Because the coefficient of v_1 is non-zero, we can solve for it, namely $v_1 = 0v_2 + \frac{1}{2}v_3$. Similarly, we can express v_3 as a linear combination of v_1 and v_2. However it is not possible to write v_2 as a linear combination of v_1 and v_3.

3.3.2. Be sure you understand WHY the matrix to reduce has the given vectors as its columns.

3.3.5d. The only vector in the null space of A^T is the zero vector and thus $N(A^T)$ does not have an independent spanning set.

3.3.7. If $\{v, w\}$ is independent, then $\{v - w, 2v + w\}$ is independent as well.

Proof. Suppose c, d are scalars satisfying $c(v-w)+d(2v+w) = 0$. Then $(c+2d)v+(d-c)w = 0$. Since v, w are independent, we conclude that $c + 2d = 0$ and $d - c = 0$. The only solution to this system of two equations in two unknowns is $c = d = 0$, which establishes the desired conclusion.

3.3.9. If $v_1, v_2, \ldots, v_k$ are non-zero and mutually orthogonal, then they are independent.

Proof. Suppose $c_1v_1 + \cdots + c_kv_k = 0$. Given i between 1 and k, we can dot both sides of this equation with v_i. This yields $c_i||v_i||^2 = 0$. Since $v_i \neq 0$, we conclude each $c_i = 0$ as desired.

3.3.12. If $k > n$, then any set of k vectors in $\mathbb{R}^n$ is dependent.

Proof. Let $v_1, \ldots v_k$ be vectors in $\mathbb{R}^n$. Form an $n \times k$ matrix A having these vectors as its columns. Since A has more columns than rows, its null space is non-trivial, which means there is a non-trivial way to express the zero vector as a linear combination of the columns of A.

3.3.19a. Suppose v_1, v_2 are non-zero vectors, and A is a matrix satisfying $Av_1 = v_1$ and $Av_2 = 2v_2$. Then v_1, v_2 are independent.

Proof. Suppose $c_1v_1+c_2v_2 = 0$. Applying A to both sides of this equation yields $c_1v_1+2c_2v_2 = 0$. Subtracting the former equation from the latter yields $c_2v_2 = 0$ Since $v_2 \neq 0$, this implies $c_2 = 0$, whence $c_1 = 0$ by substitution.