Review Session:

Field Axioms

Defn: A ordered field is a field F equipped with a distinguished subset satisfying.

1) Closure: If $x, y \in F^+$, then $x+y \in F^+$, and $xy \in F^+$
2) Trichotomy Given $x \in F$ then exactly one of the following holds a) $x \in F^+$, $-x \in F^+$, $x = 0$.

Defn: $x > y$ ($y < x$) means $x - y \in F^+$
$x \geq y$ (or $y \leq x$) means $x > y$ or $x = y$

Proposition: If $x > y$ and $y > z$. Then $x > z$.

Proof: Since $x > y$, we know $x-y \in F^+$, since $y > z$, we know $y - z \in F^+$. By closure under addition, $(x-y) + (y-z) \in F^+$. $\therefore x - z \in F^+$ $\therefore x > z$

Proposition: $1 > 0$

Proof by Trichotomy either
1) $1 \in F^+$ (fine)
2) $-1 \in F^+ \Rightarrow (-1)(-1) \in F^+ \Rightarrow 1 \in F^+$ (impossible)
3) $1 = 0$ (excluded) by field axioms.

Proposition: $\mathbb{C}$ cannot be made into an ordered field.

Proof: Suppose it could with distinguished set $\mathbb{C}^+$ $i \in \mathbb{C}$

By trichotomy, either (a) $i \in \mathbb{C}^+ = i(i) \in C^+ \Rightarrow -1 \in C^+$
(b) $-i \in \mathbb{C}^+ = -i(-i) \in C^+ \Rightarrow -1 \in C^+$
(c) $i = 0$ excluded by defn of $\mathbb{C}$

Example of an ordered fields ($\mathbb{R}$, $\mathbb{Q}$ to be discussed)

$\mathbb{Q}[\sqrt{2}] = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \}$

F ordered field, $S \subseteq F$, $a \in F$

Defn: a is an upper bound for S if $a \geq x$ for each $x \in S$.

Defn: a is a least upper bound of S if

(1) a is an upper bound of S

(2) if b is any upper Bound for S, then $a \leq b$.

Defn: F is complete if every non-empty subset of F which has an upper bound must have a least upper bound.

Ex: $\mathbb{Q}$ is not complete.

Th: $\exists$ essentially complete ordered field, which we call $\mathbb{R}$.

$S = [0,1] = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$

Upper bounds: 500, 2, 1

LUB: 1 GLB=0

$S = (0,1) = \{x \in \mathbb{R} \mid 0 < x < 1\}$

Upper bounds: 500, 2, 1

LUB: 1 is still the least upper bound

GLB = 0

Defn: $S \subseteq \mathbb{R}$ is inductive if $(\forall x \, (x \in S \Rightarrow x+1 \in S))$

example "$\mathbb{N}$", "$\mathbb{Z}$"

$[17, \infty)$, $\emptyset$, $\mathbb{R}$

Defn: $\mathbb{N}$ is the intersection of all inductive sets which contain 1.

Th: (PMI) Suppose $S \subseteq \mathbb{N}$ satisfying

i) $1 \in S$

ii) S induct.

$\therefore S = \mathbb{N}$

$S = \{5 + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$

4,	$5+\frac{1}{2}$	$5-\frac{1}{3}$,	$5+\frac{1}{4}$	4 is the smallest element of S
$n=1$	$n=2$	$n=3$	$n=4$	$5+\frac{1}{2}$ is the largest ele. of S

$S = \{5 + \frac{1}{n} \mid n \in \mathbb{N}\}$

$6, 5\frac{1}{2}, 5\frac{1}{3}, \ldots$ 5.01 when $n = 100$

6 is largest ELT, a fortiori it is the LUB

S has no smallest element. but $5 = $ GLB

Proof: (1) 5 is a lower bound of S.

(2) Suppose $b > 5$

$b - 5 > 0$

$\frac{1}{b-5} > 0 \Rightarrow 1 > b - 5$

Can find $n \in \mathbb{N}$ with $n > \frac{1}{b-5}$ by arch property

$\therefore \frac{1}{n} < b - 5$ $\quad 5 + \frac{1}{n} < b$ b is not a lower bound of S.

Defn: $f : \mathbb{N} \to \mathbb{R}$ is eventually positive if $\exists N \in \mathbb{N}$ such that $f(n) > 0$ whenever $n \geq N$.

Defn: $f : \mathbb{N} \to \mathbb{R}$ diverges to ∞ if given $M \in \mathbb{R}$, can find $N \in \mathbb{N} \ni f(x) - M > 0$ whenever $n \geq N$

notation $\lim_{n \to \infty} f(n) = \infty$.

Defn: $L \in \mathbb{R}$ $f : \mathbb{N} \to \mathbb{R}$ (sequences) f converges to L if given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ such that $|f(n) - L| < \varepsilon$ whenever $n \geq N$.

Defn: $f : \mathbb{R} \to \mathbb{R}$ $\lim_{x \to a} f(x) = L$ $(a, L \in \mathbb{R})$

If given $\varepsilon > 0$, can find $\delta > 0 \ni |f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x \to 2} x + 2 = 4$

Defn: $\lim_{x \to \infty} f(x) = L$ $(L \in \mathbb{R})$

Given $\varepsilon > 0$, $\exists k$, such that $|f(x) - L| < \varepsilon$ whenever $x \leq k$

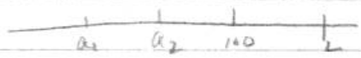
notation: $(a_n)_{n=1}^{\infty}$ Can define sequences
1) By Formula
2) Recursively

Examples:

1) Fibonacci Number
2) Partial Sums
3) Newton Methods
4) Factorials

Can Prove sequence converge from the def'n:
Algebraic Properties

Th: Every BDD increasing sequence to its least upperbound.
Proof: $\mathcal{S}$ (a_n) is increasing & bounded (above)
by defn $\exists M \geq a_n \; \forall n \in \mathbb{N}$ i.e. M is an upper bound of
$S = \{a_n \mid n \in \mathbb{N}\}$
Write $L = LUB(S)$
Claim $a_n \to L$



Let $\varepsilon > 0$ be given. Since L is an upper bound of S
at least have $a_n \leq L < L + \varepsilon$ for all n.
Since L is the least upper bound of S, $L - \varepsilon$ cannot
be an upper bound of S. Thus $\exists N \in \mathbb{N}$ with
$a_N > L - \varepsilon$
Suppose $n \geq N$, then $a_n \geq a_N$. Then $L + \varepsilon > L \geq a_n \geq a_N > L - \varepsilon$
$-\varepsilon < a_n - L < \varepsilon \;\; \therefore |a_n - L| < \varepsilon$

Th: Every bounded sequence has a convergent subsequence.
Th: Every cauchy sequence converges

<u>Defn</u>: The Infinite series $\sum_{n=1}^{\infty} a_n$ <u>converges</u> to some number L
If its sequence (S_n) of partial sums converges to L.

Can use defn directly for ① geometric series + ② Telescoping series.

Test: (1) n'th term
(2) regular comparison
(3) Limit Comparison
(4) Ratio Test
(5) Integral
(6) Absolute. If $\sum |a_n|$ conv. Then $\sum a_n$ conv.
(7) Alt. series Test

Particular Series: ① $\sum \frac{1}{n}$ div. $\sum \frac{1}{n^2}$ conv. $\sum \frac{1}{n^p}$ $\begin{cases} \text{conv. when } p>1 \\ \text{div when } p<1 \end{cases}$

(Integral Test)

② $\sum r^n$ converges to $\frac{1}{1-r}$ for $|r|<1$
diverges otherwise

$\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \Rightarrow \lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n \ln\left(1+\frac{1}{n}\right)$

$\lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{1}{n}}\right)\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-\frac{1}{n^2}\right)}} = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right) = 1$

$e^1 = e$

From the defn. $\left[\begin{array}{l} \lim_{x\to 2} 2x+5 = 9 \\ \lim_{x\to 2} x^2+3x = 10 \end{array}\right.$

$f(x) = x^2+3x$

$L = 10$

$a = 2$

$|f(x)-L| = |x^2+3x-10| = |x+5|\,|x-2|$ want $< \varepsilon$

Assume $|x-2|<1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3 \Rightarrow$

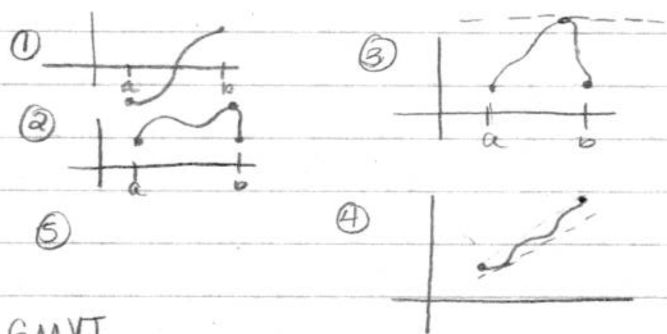
$6 < x+5 < 8 \quad |x+5| < 8$

Let $\varepsilon > 0$ be given. Take $\delta = \min(1, \varepsilon/8)$, if $|x-2| < \delta$
then $|x-2| < 1$ and $|x-2| < \varepsilon/8$

$|f(x)-L| = |x^2-3x-10| = |x+5|\,|x-2| < 8\,|x-2| < 8\left(\frac{\varepsilon}{8}\right) = \varepsilon$

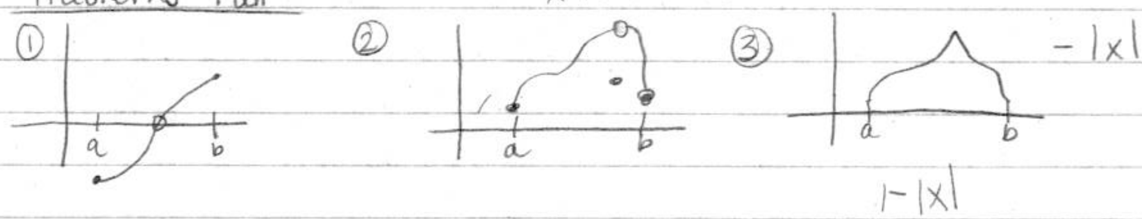
Theorems:

1) Intermediate Value
2) Maximum Value
3) Rolle's Theorem
4) MVT
5) GMVT



$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)} \text{ GMVT} \qquad MVT = \frac{f(b)-f(a)}{b-a}$$

Theorems Fail: $\quad$ -X if not a closed interval



Defn's: TP, remainder, T.S.

Taylors Theorem: Suppose f has derivatives of all orders on some interval $I = (a-\varepsilon, a+\varepsilon)$ $x \in I$.

P_n = n'th degree Taylor Polynomial of f about a.

r_n = n'th remainder = $f(x) - P_n(x)$

Then $P_n(x)$ is a good approximation of $f(x)$ in 3 senses

1) at a: $r_n(a) = r_n'(a) = \ldots = r_n^{(n)}(a) = 0$

2) near a: $\lim\limits_{x \to a} \frac{r_n(x)}{(x-a)^n} = 0$

3) error quant: $r_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ some c between x + a

Approx: sin(10) to five decimal places

Sol'n: $f(x) = \sin x \qquad x = 10 \qquad a = 0$

$$r_n(x) = \left| \frac{f^{(n+1)}(c)\,(x)^{n+1}}{(n+1)!} \right| \overset{\text{have}}{\leq} \frac{10^{n+1}}{(n+1)!} \overset{\text{want}}{<} 10^{-5}$$

i.e. $\frac{10^{n+1}}{n+1} < \frac{1}{10^5}$ $\qquad$ i.e. $10^{n+6} < (n+1)!$

n	10^{n+6}	$(n+1)!$
|	|	|
|	|	,

work this out until its satisfying $10^{n+6} < (n+1)!$

$\ln(1.1)$

$f(x) = \ln(x) \quad x = 1.1$

$f'(x) = x^{-1} \qquad a = 1$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(n+1)}(x) = (-1)^n (n!)\, x^{-(n+1)}$

$$R_n(x) = \left| \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!} \right| = \left| \frac{n!(c)^{-(n+1)}(.1)^{n+1}}{(n+1)!} \right| =$$

$$= \left| \frac{(.1)^{n+1} \frac{1}{c^{n+1}}}{n+1} \right| \le \frac{(.1)^{n+1}}{n+1}$$

$$\sum_{n=0}^{\infty} x^n \sim \frac{1}{1-x}$$

$$\sum_{n=0} t^n = \frac{1}{1-t} \qquad \int_0^x \left(\sum_{n=0}^{\infty} t^n\right) dt \sim \int_0^x \frac{1}{1-t}\, dt$$

$$\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} \Big|_0^x \sim -\ln(1-t) \Big|_{t=0}^{t=x} \qquad \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} \sim -\ln(1-x)$$

$$\sum_{n=0}^{\infty} \frac{(x^3)^{n+1}}{n+1} \sim -\ln(1-x^3)$$

$$\sum_{n=0}^{\infty} \frac{x^{3n+3}}{n+1} \sim -\ln(1-x^3) \qquad \sum_{n=0}^{\infty} \frac{x^{3n}}{n+1} \sim -\frac{\ln(1-x^3)}{x^3}$$

1) Find domain of Convergence of a Power Series.
2) Uniform vs. Pointwise conv.

$f_n(x) = \frac{x^n}{n}$ on $[0,1]$

$g_n(x) = x^n$ on $[0,1]$

① converges uniformly to 0

② Does not converges uniformly

Find upper bound that don't depend on x

f_n converges uniformly to 0 on D.

Given $\varepsilon > 0$ $\exists N \in \mathbb{N}$ s.t. when $x \in D$ & $n \geq N$ then $|f_n(x)| < \varepsilon$

Scratch: $|f_n(x)| \leq \frac{1}{n} < \varepsilon$ $\quad N > \frac{1}{\varepsilon}$

Given $\varepsilon > 0$. Take $N \in \mathbb{N}$ s.t. $N > \frac{1}{\varepsilon}$. Suppose $n \geq N$ and $x \in [0,1]$ Then $\left|\frac{x^n}{n}\right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

Negation: g_n does not converge converge uniformly to 0.

$\exists \varepsilon > 0$ s.t. $\forall N \in \mathbb{N}$ $\exists x \in D$ and $n \geq N$ such that $|f_n(x)| \geq \varepsilon$

Let $\varepsilon = \frac{1}{2}$. Let N be given. Take $n = N$ and $x = \sqrt[n]{\frac{1}{2}}$.

$|f_n(x)| = |((\frac{1}{2})^{1/n})^n| = \frac{1}{2} = \varepsilon$.