Sequences and Series

Notes for MATH 3100 at the University of Georgia Spring Semester 2005

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Introduction

1. Overview

Intuitively, a *sequence* is an unending list of numbers. The primary problem in the subject is to decide whether a given sequence *converges* in the sense that the entries on the list approach a fixed number. A little experimentation should convince you that the sequences

$$1,\ 1,\ 1,\ 1,\ \ldots \tag{0.1}$$

$$1,\ 0.1,\ 0.01,\ 0.001,\ \ldots \tag{0.2}$$

$$1,\ 1.1,\ 1.11,\ 1.111,\ \ldots \tag{0.3}$$

converge to 1, 0, and $\frac{10}{9}$ respectively.

On the other hand, the sequences

$$1,\ -1,\ 1,\ -1,\ \ldots \tag{0.4}$$

$$1,\ 2,\ 3,\ 4,\ \ldots \tag{0.5}$$

diverge in the sense that neither converges to any number.

Informally, a *series* is an expression consisting of numbers separated by plus signs, like

$$1\ +\ 0.1\ +\ 0.01\ +\ 0.001\ +\ \ldots. \tag{0.6}$$

The numbers appearing in a series are called *terms* and can be arranged in a sequence. Thus (2) is the sequence of terms associated with the series (6). A more important sequence associated with a series is the sequence of *partial sums* formed by keeping a record of successive cumulations of the terms. A series is said to *converge* if its sequence of partial sums converges. The sequence of partial sums of (6) is given by (3), so we see that the series (6) converges to $\frac{10}{9}$ and it is customary to write

$$1\ +\ 0.1\ +\ 0.01\ +\ 0.001\ +\ \ldots\ = \frac{10}{9} \tag{0.7}$$

A more interesting series is

$$(0.1)^1\ +\ (0.1)^{1+2}\ +\ (0.1)^{1+2+3}\ +\ (0.1)^{1+2+3+4}\ +\ \ldots \tag{0.8}$$

It is tempting to guess that this series converges to $0.1010010001\ldots$, but this is just begging the question because we don't really know what such an infinite decimal expansion means. This dictates the starting point of our course as a careful discussion of the real number system, to be given in Chapter 1. In Chapter 2, we will give precise definitions of sequences and their limits, and learn some shortcuts for dealing with the limit concept. This will be applied in Chapter 3, where we will develop an arsenal of techniques for deciding which series converge.

Chapter 4 represents a slight detour from the study of series; it will apply sequences to the study of continuous functions.

Chapters 5 will be devoted to the study of series of *functions.* To make the transition, we generalize Equation 0.7, noting that

$$(0.9)\qquad 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}, \quad \text{whenever } -1 < x < 1.$$

This is called a *power series representation* of the function $f(x) = \frac{1}{1-x}$. In Chapter 5, we will learn Taylor's Theorem — a tool which makes it painless to find power series representations for functions like e^x and $\sin x$. This in turn makes it easy to approximate numbers like $\sqrt{e}$ — by hand and to any desired degree of accuracy!

In Chapter 6, we will adapt much of our earlier work to sequences and series of *complex* numbers. This will serve as a brief introduction to the complex variables course (MATH 4150) which many of you will soon take. In particular, consideration of complex power series will take the mystery out of de Moivre's formula

$$\mathrm{e}^{ix} = \cos x + i \sin x$$

which you have probably already used in differential equations.

The book closes with outlines of several ways to construct the real number system from the more familiar rational numbers.

2. Prerequisites

One of the purposes of MATH 3100 is to smooth the transition between "computational" calculus courses and "rigorous" 4000–level courses. Other "transitional" courses in the Department are MATH 3000 (Linear Algebra) and MATH 3200 (Introduction to Higher Mathematics). The curriculum was designed so that these three courses, along with MATH 2500 and MATH 2700, can be taken in any order; their common prerequisite is MATH 2210 (Integral Calculus).

This is perhaps a good place to point out just what we will need from calculus. Of most immediate use will be intuitive feeling for and ability to compute limits; L'Hôpital's rule will often come in handy. You should also be familiar with exponential, logarithmic, trigonometric, and inverse trigonometric functions, and the basic differentiation and integration techniques. We probably won't be doing any fancy trig substitutions, but we will definitely use the fact that $\int \frac{1}{1+x^2} dx = \arctan x$, and may need an occasional partial fraction decomposition toward the end of the course.

3. References

The following sources were used in constructing notes for our course. (MATH 3100 was MAT 350 under quarters.)

(1) MAT 350 notes by Kevin F. Clancey
(2) MAT 350 notes by David E. Penney
(3) Calculus, by Michael Spivak, 2nd Edition, 1967, Publish or Perish, Houston.
(4) Principles of Mathematical Analysis, by Walter Rudin, 3rd Edition, 1976, McGraw Hill, New York.

Please do not hesitate to ask questions concerning these course notes either in or out of class. Corrections and expository suggestions are also welcome.

4. Notations

The following are some common mathematical notations that will be used frequently when writing on the board to save time:

- $\forall$ for any. Example: $\forall$ real number x, $x^2 \geq 0$.
- $\exists$ there exists. Example: $\exists$ a real number x with $x^2 < 1$
- $x \in F$ x is an element of the set F. Example: $2 \in \mathbb{Q}$.
- $x \notin F$ x is *not* an element of the set F. Example: $\sqrt{2} \notin \mathbb{Q}$.
- $E \subset F$ the set E is a subset of the set F, i.e. every element of E is also an element of F: $\forall x \in E$, $x \in F$. Example: $\mathbb{Q} \subset \mathbb{R}$.
- $E \not\subset F$ the set E is *not* a subset of the set F: $\exists x \in F$ such that $x \notin E$. Example: $\mathbb{R} \not\subset \mathbb{Q}$.
- $\square$ end of proof.

5. Acknowledgements

I would like to thank several people for contributing to the development of these notes. Drs. Clancey and Penney let me examine their own notes and gave me the benefit of their experience with the course. Drs. Alexeev, Benson, and Rumely provided valuable feedback on earlier versions of the notes. In particular, Dr. Benson added the section on Dirichlet's test, while Dr. Alexeev added the decimal-based construction of the real numbers, and transformed my original AMSTeX file to the more flexible AMSLATeX format.

I'm also grateful for numerous student suggestions over the years, especially Evan Glover's efforts to the make whole enterprise more "student friendly".

CHAPTER 1

Real Numbers

Intuitively, real numbers are the "measure numbers", with which one can measure arbitrary distances. The usual model all of us have in mind is that of a real line, with a chosen origin and unit measure. In this model, a real number corresponds to a point on the line. However, this picture relies on properties of the physical world. What is a point on a line? How does a line look when you look at it at a greater and greater magnification? Clearly, "point" and "line" are mathematical abstractions of the physical world, and have to be dealt with mathematically.

There are various ways to study the real number system. The "bottom-up" approach starts out with a set of axioms for the set of natural numbers $\{1, 2, 3, ...\}$, and then constructs successively larger number systems until the full real number system is reached. This is the approach taken in Dr. Clancey's notes, and one you may see in MATH 4000. It makes a strong case for the existence of the real number system, but it is somewhat technical and takes a good deal of time to complete. We therefore postpone this discussion till the last chapter of the book.

In this chapter, we will take the "top-down" approach: we begin by discussing various properties that we want the full real number system to enjoy. The goal is a minimal set of axioms which characterize the real number system. We then "look down" to find natural numbers, integers and rational numbers inside the reals. While not quite as logically satisfying as the bottom–up approach, this procedure quickly sets the tone for the type of reasoning we will be using throughout the course. Several implementations of the "bottom-up" approach are outlined in Chapter 7.

1. Fields

DEFINITION 1.1. A *binary operation* on a set S is a function $f : S \times S \to S$. We usually write the name of the function between its arguments.

DEFINITION 1.2. A *field* is a set F equipped with two binary operations, denoted "+" and $\cdot$ satisfying the following properties:

Axioms for addition:

- A1 (closure) If $x \in F$ and $y \in F$, then $x + y \in F$.
- A2 (commutativity) If $x \in F$ and $y \in F$, then $x + y = y + x$.
- A3 (associativity) If x, y, and z each belong to F, then $(x + y) + z = x + (y + z)$.

A4 (neutral element) There is an element $0 \in F$ such that $0 + x = x + 0 = x$ for every element $x \in F$.
A5 (inverses) Given $x \in F$, there is an element $-x \in F$ satisfying $(-x) + x = x + (-x) = 0$.

Axioms for multiplication:

M1 (closure) If $x \in F$ and $y \in F$, then $x \cdot y \in F$.
M2 (commutativity) If $x \in F$ and $y \in F$, then $x \cdot y = y \cdot x$.
M3 (associativity) If x, y, and z each belong to F, then $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.
M4 (neutral element) There is an element $1 \in F$ such that $1 \cdot x = x$ for every element $x \in F$. It is assumed that $1 \neq 0$.
M5 (inverses) Corresponding to each *non–zero* member x of F, there exists an element $x^{-1} \in F$ satisfying $x \cdot x^{-1} = 1$.

Distributive law: If x, y, z each belong to F, then $x \cdot (y + z) = x \cdot y + x \cdot z$.

A moment's thought should convince you that we certainly want the real number system to be a field. Is there anything else we want? The next few propositions show that many properties of the reals we "forgot" to mention are consequences of what we already have.

Our primary goal at this point is an appreciation of the power of the field axioms. More detail will be left for later courses (MATH 4000 and MATH 4100), and we will soon resume taking such "obvious" facts for granted. In the meantime, we will complete a few sample proofs in class and in the homework. First, we recall some common abbreviations.

for	**write**
$x \cdot y$	xy (suppress the multiplication sign)
$x + (-y)$	$x - y$
xy^{-1}	$\frac{x}{y}$ (justified by commutativity)
$(x + y) + z$	$x + y + z$
$(xy)z$	xyz (justified by associativity)
$x + x$	$2x$ (etc.)
xx	x^2 (etc.)
$(x^{-1})^2$	x^{-2} (etc.)

Proposition 1.3. *The addition axioms imply the following for any $x, y, z \in F$:*

(1) *(cancellation) If $x + y = x + z$, then $y = z$.*
(2) *(uniqueness of neutral element) If $x + y = x$, then $y = 0$.*
(3) *(uniqueness of inverse) If $x + y = 0$, then $y = -x$.*
(4) *(double negative) $-(-x) = x$.*

Proof. Of course, the way to establish (1) is to add $-x$ to both sides of the given equation. Axiom A5 assures us that $-x$ exists and since binary operations are functions, we know that "equals added to equals are equal". Thus we have

$$(-x) + (x + y) = (-x) + (x + z). \tag{1.1}$$

Working on the left-hand side of this equation, we apply Axioms A5, A3, and A4 in turn to obtain $(-x)+(x+y) = ((-x)+x)+y = 0+y = y$. Similarly, the right hand side of Equation 1.1 simplifies to z, and we conclude $y = z$ as desired.

We could repeat the preceding argument to establish (2), but it is a little neater to apply the *result* of (1). Indeed, putting Axiom A4 and the hypothesis of (2) together, we have $x + y = x + 0$, and thus (1) yields $y = 0$ as desired. The proof of (3) is similar.

Finally, for (4) write $y := -x$. Because y is the additive inverse of x, we have $x + y = 0$. But this equation also tells us that x is the additive inverse of y, i.e., $x = -y = -(-x)$as desired. □

Slight variation in the wording of this proof gives the corresponding facts for multiplication.

PROPOSITION 1.4. *The multiplication axioms imply the following for any* $x, y, z \in F$*:*

(1) *(cancellation) If* $x \neq 0$ *and* $xy = xz$*, then* $y = z$*.*
(2) *(uniqueness of neutral element) If* $x \neq 0$ *and* $xy = x$*, then* $y = 1$*.*
(3) *(uniqueness of inverse) If* $xy = 1$*, then* $y = x^{-1}$*.*
(4) *(double inverse) If* $x \neq 0$ *then* $(x^{-1})^{-1} = x$*.*

The next result is more subtle because it *simultaneously* involves multiplication and additon. This dictates the use of distributivity as that is the only axiom involving both operations.

PROPOSITION 1.5. *The field axioms have the following consequences for any* $x, y \in F$*.*

(1) $0x = 0$.
(2) $xy = 0$ *implies* $x = 0$ *or* $y = 0$*.*
(3) $(-x)y = -(xy) = x(-y)$*.*
(4) $(-x)(-y) = xy$*.*

PROOF. For (1), we apply distributivity to conclude $(0+0)x = 0x+0x$. Since $0 + 0 = 0$, we get $0x + 0x = 0x$, whence $0x = 0$ by Proposition 1.3(2).

To establish (2), we assume that $xy = 0$, while $x \neq 0$. In view of (1) and commutativity, we have $xy = x0$,whence $y = 0$ by multiplicative cancellation.

For (3), we combine distributivity and Part (1) to get $(-x)y + xy = ((-x) + x)y = 0y = 0$, whence $(-x)y$ must be the additive inverse of xy by Proposition 1.3(3).

Applying (3) with $-y$ playing the role of y yields
$(-x)(-y) = -(x(-y)) = -(-(xy))$. Since this simplifies to xy by the double negative property, we have (4) and the proof is complete. □

In particular, now you finally know why $(-1)(-1) = 1$. Several exercises at the end of the chapter are designed to bring home the centrality of these observations in school mathematics.

The following embarrassing example shows that there must be more to the real number system than the field axioms.

EXAMPLE 1.6. Take F to be the set whose only elements are 0 and 1. Define addition on F by $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$. Define multiplication on F via the equations $0(1) = 1(0) = 0(0) = 0$ and $1(1) = 1$. An easy but boring check shows that this F is a field.

Since there are more than two real numbers, we must be missing something.

2. Order

What we need is order. It turns out to be convenient to first axiomitize the notion of positivity.

DEFINITION 1.7. An *ordered field* is a field F equipped with a distinguished subset F^+ satisfying the following.

(1) (trichotomy) If $x \in F$, then one and only one of the three statements $x \in F^+$, $x = 0$, $-x \in F^+$ is true.
(2) (closure) If x, y belong to F^+, then $x+y$ and xy also belong to F^+.

Members of the distinguished set F^+ are called *positive*.

The following proposition highlights the subtle power of the trichotomy law.

PROPOSITION 1.8. *Let F be an ordered field with positive set F^+.*

(1) $x^2 \in F^+$ *for each non–zero member x of F.*
(2) $1 \in F^+$.
(3) *If $x \in F^+$, then x^{-1} also belongs to F^+.*

PROOF. For (1), we apply the trichotomy law to x. Since $x \neq 0$ by hypothesis, there are really only two possibilities. If $x \in F^+$, then $x^2 \in F^+$ by closure. On the other hand, if $-x \in F^+$, then we get $x^2 = (-x)^2 \in F^+$ by Part (4) of Proposition 1.5 and closure.

Part (2) follows from Part (1) since we know that $1 = 1^2$.

For Part (3), we apply trichotomy to x^{-1}. Having $x^{-1} = 0$ would contradict Part (1) of Proposition 1.5. Having $-x^{-1} \in F^+$ would force $-1 = x(-x)^{-1} \in F^+$ by Part (3) of Proposition 1.5 and closure, thereby contradicting (2). Thus we have eliminated all possibilities *except* $x^{-1} \in F^+$. □

Note that $1 + 1$ must be a positive member of any ordered field, so the field of Example 1.6 *cannot* be ordered. More generally, 0, 1, $1 + 1$, $1 + 1 + 1$, etc. must be distinct in any ordered field F — intuitively F must contain a "copy" of the natural numbers $1, 2, 3, 4, \ldots$. In fact, the field axioms concerning inverses show that F must contain a "copy" of the rational number system.

It is time to adopt the usual inequality notation associated with ordered fields. Both $y > x$ and $x < y$ mean $y - x \in F^+$; we also write $y \geq x$ as an abbreviation for $y > x$ or $y = x$. We say x is *positive* if $x > 0$ and we call x *non–negative* when $x \geq 0$. All the familiar rules for working with inequalities follow from what we've done so far. Once again, we practice with a few formal proofs. (You can now use results of Propositions 3, 4, and 5 above without comment).

PROPOSITION 1.9. *Let x, y, and z be members of an ordered field F.*

(1) *(unique comparability) Exactly one of the following 3 statements is true:* $x < y, \quad x = y, \quad y < x.$
(2) *(transitivity) If $x < y$ and $y < z$, then $x < z$.*
(3) *If $x < y$, then $x + z < y + z$.*
(4) *If $z > 0$ and $x < y$, then $xz < yz$.*
(5) *If $z < 0$ and $x < y$, then $xz > yz$.*
(6) *If $0 < x < y$, then $y^{-1} < x^{-1}$.*

PROOF. For (1), we apply the trichotomy axiom to $y - x$. Indeed the mutually exclusive possibilities $y - x \in F^+$, $y - x = 0$, and $-(y - x) \in F^+$ directly correspond to $x < y$, $x = y$, and $x > y$ respectively.

For (2), we apply the apply the definition of $<$ to conclude that both $y - x$ and $z - y$ belong to F^+. Thus their sum $z - x$ must also belong to F^+, and that means that $x < z$ as desired. For (3), it suffices to note that $(y + z) - (x + z) = y - x$.

For (4), we apply the definition to get z and $y - x$ in F^+. Thus their product $yz - xz \in F^+$, so $xz < yz$ as desired. The proof of (5) is similar, except that we begin by noting that $-z$ and $y - x$ both belong to F^+. Finally, we get (6) by applying (4) with $z = x^{-1}y^{-1}$. □

The properties of Proposition 1.9 can be used to design as well as solve inequalities. The following example will come up in the next chapter.

EXAMPLE 1.10. Compare the expressions $3n - 10000$ and n.

SOLUTION. Let $\sim$ stand for one of the signs "$>=, <$". Then $3n - 10000 \sim n$ if and only if $n \sim 5000$. In particular, $3n - 10000 > n$ whenever $n > 5000$. □

3. Absolute Value

Absolute values will play an important role in the sequel. The intuitive idea is that $|x - y|$ should measure the *undirected* distance between x and y.

Definition 1.11. Let x be a member of an ordered field F. Then the *absolute value* of x, denoted $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}.$$

Example 1.12. The equation $|x| = 5$ has two solutions, namely $x = 5$ and $x = -5$. On the other hand, the equation $|x| = -2$ has no solutions.

Consider next the equation $2x + |x - 3| = 5$. We deal with the absolute value by considering cases.

(1) $x - 3 \geq 0$. Then $|x - 3| = x - 3$ by definition. Substituting in our target equation gives $2x + (x - 3) = 5$, i.e., $x = \frac{8}{3}$. But this is a fake solution as it doesn't satisfy the target equation. (Also $\frac{8}{3}$ is outside the interval defining this case.)
(2) $x - 3 < 0$. Here the defintion yields $|x - 3| = -(x - 3) = 3 - x$. Substituting in our target equation gives $2x + (3 - x) = 5$, i.e., $x = 2$, which does indeed satisfy the target equation.

In summary, $x = 2$ is the only solution of the equation $2x + |x - 3| = 5$.

Proposition 1.13. *Let x and y be members of an ordered field.*

(1) $|xy| = |x||y|$.
(2) $|x|$ *is the maximum of* x *and* $-x$.
(3) $|x| < y$ *if and only if* $-y < x < y$.
(4) *(triangle inequality)* $|x + y| \leq |x| + |y|$.

Proof. We use results of earlier propositions without comment. The straightforward consideration of cases needed for Part (1) is left to the reader.

For (2) we consider two cases. When $x \geq 0$, we have $-x \leq 0 \leq x$, so $|x| = x$ is indeed the maximum of $x, -x$. Similarly, when $x < 0$, then $-x > 0 > x$, so $|x| = -x$ is still the maximum of $x, -x$.

In view of (2), the inequality $|x| < y$ is equivalent to requiring both $x < y$ and $-x < y$. Since the latter two inequalities are in turn equivalent to the double inequality $-y < x < y$, we have established (3).

Applying (2) to x and y individually, we obtain

$$x \leq |x|, \quad y \leq |y|, \quad -x \leq |x|, \quad \text{and} \quad -y \leq |y|.$$

Adding these in pairs gives

$$x + y \leq |x| + |y|, \quad \text{and} \quad -(x + y) \leq |x| + |y|.$$

whence another appeal to Part (2) establishes (4). □

Example 1.14. By Part (3) of the preceding proposition, the inequality $|x - 3| < 5$ is equivalent to $-5 < x - 3 < 5$ and thus its solution set is the open interval $(-2, 8)$. Note that this corresponds to all points on the number line obtained by wandering at most 5 units from 3.

The solution of the opposite inequality $|x-3| \geq 5$ is the complementary set, namely the union $(-\infty, -2] \cup [8, \infty)$.

EXAMPLE 1.15. How large can $|x+6|$ be when $|x-3| < 1$?

In view of (3) of the preceding proposition, the given condition means $-1 < x - 3 < 1$. Adding 9 then yields $8 < x + 6 < 10$, which shows $|x+6| < 10$.

A more sneaky approach is to use the triangle inequality to conclude $|x+6| = |(x-3)+9| \leq |x-3| + |9| < 10$.

EXAMPLE 1.16. Consider the polynomial $p(x) = x^2 + 3x$. Then $p(3) = 18$ and it seems reasonable that $p(x)$ should be close to 18 when x is close to 3. To quantify this, note that

$$|p(x) - 18| = |x^2 + 3x - 18| = |x+6||x-3|.$$

In view of the last example, we have

$$|p(x) - 18| = |x+6||x-3| \leq 10|x-3|, \qquad \text{whenever } |x-3| < 1.$$

From this it follows for example that $|p(x) - 18| < .01$ whenever $|x-3| < .001$.

4. Completeness

To finish our characterization of the real number system, we must understand how it differs from the rational number system. We will see shortly that there is no rational solution to the equation $x^2 = 2$. It's not that we can't get close: the squares of $1, 1.5, 1.41, 1.414, \ldots$ get closer and closer to 2. What we need is a "limit" of these numbers; our last axiom will guarantee that such a limit exists.

DEFINITION 1.17. Let S be a subset of an ordered field F.

(1) An element $b \in F$ is an *upper bound* of S if $x \leq b$ for all $x \in S$.
(2) We call b the *least upper bound* of S if in addition every upper bound a for S satisfies $b \leq a$.
(3) An ordered field F is said to be *complete* if every non-empty subset of F which has an upper bound must have a least upper bound as well.

It will probably take some time for you to feel comfortable with these concepts; that's one of the goals of the course. As a first step, plot the sets discussed in the following (informal) example on a number line.

EXAMPLE 1.18. Take F to be the rational numbers.

(1) If $S = \{x \in F : x < 3\}$, then 3, 4, and 543 are all upper bounds for S, and 3 is its least upper bound. Note that $\operatorname{lub} S \notin S$.
(2) If $S = \{x \in F : 0 < x \text{ and } x^2 < 2\}$, then $2, 1.5, 1.42, \ldots$ are upper bounds of S, but the irrationality of $\sqrt{2}$ means that S cannot have a lub.

(3) Take $S = \{.1, .101, .101001, .1010010001, \dots\}$. Again, there are many upper bounds to this set, but the non–repeating nature of these decimal expansions rules out the existence of a *rational least* upper bound for S.

No ordered field has an upper bound, much less a least upper bound. The point of the preceding example is that some **bounded** sets of rational numbers have least upper bounds and others don't. This means that the rational number field is not complete. Geometrically, not all points on a line can be accounted for by plotting rational distances.

There are several competing notions of completeness in mathematics, but in this course "complete" will always be used in the sense of Definition 1.17. Hopefully, you are convinced that the real number system should be a complete ordered field, but several questions remain.

(1) How do we know that there are any complete ordered fields?
(2) How different can two complete ordered fields be?
(3) Have we left anything out in our characterization of the real number system?

Question 1 is addressed by constructing the reals from more primitive systems. We will return to this topic in Chapter 7, to which you can skip ahead now if your curiosity has gotten the best of you.

As for Question 2, we could obviously call numbers by different names - their French names for example - but the resulting number **system** would really be the same. Thus you don't have to learn a new set of "tables" to do arithmetic in French: to compute deux fois quatre, you could first translate to the English problem two times four, next do the computation in English to get eight, and finally translate the answer back to get the French huit. The second part of the following theorem states that any two complete ordered fields are related in this way. Think of the function f (technically known as an isomorphism) as a dictionary.

THEOREM 1.19. *(without proof)*

(1) *Complete ordered fields exist.*
(2) *Suppose both R and R' are both complete ordered fields. Then there is a one-to-one, onto function $f : R \to R'$ which respects all ordered field structures, e.g., $f(x + y) = f(x) +' f(y)$ for all $x, y \in R$; here $+$ denotes addition in R , while $+'$ denotes addition in R'.*

DEFINITION 1.20. The *real number system* is a fixed complete ordered field. It is denoted $\mathbb{R}$.

The essential uniqueness of the real number system is also reassuring concerning Question 3 - we have successfully characterized the system.

5. Induction

Our next task is to identify the natural, integral, and rational number systems as subsets of $\mathbb{R}$ (the "down" part of our top–down approach).

DEFINITION 1.21. (1) A subset S of $\mathbb{R}$ is *inductive* if $x+1 \in S$ whenever $x \in S$.

(2) The set of *natural numbers* is the intersection of all inductive subsets of $\mathbb{R}$ which contain 1. It is denoted $\mathbb{N}$.
(3) The set of *integers*, denoted $\mathbb{Z}$, is defined by $\mathbb{Z} = \{x \in R : x = m-n$ for some $m, n \in \mathbb{N}\}$.
(4) The set of *rational* numbers , denoted $\mathbb{Q}$, is defined by $\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$ with $n \neq 0\}$.

The interval $[1, \infty) = \{x \in \mathbb{R} | x \geq 1$ is an inductive set containing 1. Since $\mathbb{N}$ is the intersection of *all* such sets, we see that $\mathbb{N} \subset [1, \infty)$. In particular, every natural number is positive and 1 is the smallest member of $\mathbb{N}$. A more subtle inductive set is $\{x \in \mathbb{R} | x = 1$ or $x \geq 1+1\}$. Since $\mathbb{N}$ must be contained in this set as well, we see that there is no natural number between 1 and 2.

It is an exercise to show that $\mathbb{N}$ is an inductive set. Since $1 \in \mathbb{N}$, it follows that $1+1 = 2 \in \mathbb{N}$, whence $3 \in \mathbb{N}$, etc; since $\mathbb{N}$ is the smallest inductive subset of $\mathbb{R}$, we see that we really have captured the natural number system. The next Proposition shows that the principle of mathematical induction is built into the definition of $\mathbb{N}$.

PROPOSITION 1.22. *(Principle of Mathematical Induction) Suppose S is an inductive subset of $\mathbb{N}$ containing 1. Then $S = \mathbb{N}$.*

PROOF. We have $S \subset \mathbb{N}$ by hypothesis. The opposite inclusion $\mathbb{N} \subset S$ follows since $\mathbb{N}$ is the intersection of all inductive sets containing 1. □

The Principle of Mathematical Induction will play an important role in our course. The following application is typical.

EXAMPLE 1.23. Use induction to prove that $1+3+5+\cdots+(2n-1) = n^2$ for every natural number n.

SOLUTION. Set $S = \{n \in \mathbb{N} : 1+3+5+\cdots+(2n-1) = n^2\}$. Note first that $1 \in S$ since $1 = 1^2$. To see that S is inductive, suppose that $k \in S$. Then $1+3+5+\cdots+(2k-1) = k^2$. Adding $2[k+1]-1$ to both sides of this equation yields

$$1+3+5+\cdots+(2k-1)+(2[k+1]-1) = k^2+2k+1 = (k+1)^2$$

which means that $k+1 \in S$. Proposition 1.22 allows us to conclude that $S = \mathbb{N}$ and thus completes the proof. □

The discussion in the following example is somewhat naive. Full justification can be found in Section 1-8 of <u>Topology</u>, by James R. Munkres, 1975, Prentice-Hall, Engelwoods, N.J.

EXAMPLE 1.24. Induction can be used to *define* functions on the natural numbers. (This procedure is also known as *recursion*). To illustrate with factorials, set $1! = 1$ and $(n+1)! = (n+1)n!$. The set S of natural numbers whose factorials are defined by this prescription includes 1 and is inductive. It follows from Proposition 1.22 that the factorials of all natural numbers are thus defined. Use of this procedure to compute 5! helps explain why induction is the mathematical version of the "domino theory".

We proceed to our first application of completeness of $\mathbb{R}$. Proposition 1.25 will be used frequently in the sequel.

Proposition 1.25. *The set $\mathbb{N}$ of natural numbers does not have an upper bound.*

Proof. We argue by contradiction. If $\mathbb{N}$ had an upper bound, it would have a least upper bound b. But then $b - 1$ could not be an upper bound of $\mathbb{N}$, so there would be a natural number $n > b - 1$. This however implies that $n + 1 > b$, contradicting the fact that b was supposed to be an upper bound of $\mathbb{N}$. □

Corollary 1.26. *(Archimedean Principle) Suppose x and y are positive real numbers. Then there is a natural number n satisfying $nx > y$.*

Proof. The preceding proposition provides a natural number $n > \frac{y}{x}$. □

There is a folk version of the Archimedean Principle: "every little bit helps". No matter how small x is, if you keep progressing by that amount, there is no limit on how high you can go.

We continue with the promised proof that $\sqrt{2}$ cannot be rational. The argument illustrates the more informal approach to the real numbers which will be taken in later chapters. Field properties are used without comment. More significantly, the possibility of reducing fractions to lowest terms is taken for granted, even though the justification of this procedure ultimately depends on a generalized version of induction not discussed in the text. (See Problem 1.50).

Proposition 1.27. *The equation $x^2 = 2$ does not have a rational solution.*

Proof. We argue by contradiction, assuming that $x^2 = 2$ for some rational x. By definition of rational, we may write $x = \frac{p}{q}$ for some integers p, q. We may assume that this fraction is in lowest terms. Algebraic manipulations lead to the equation $2q^2 = p^2$. This implies that p^2 is even, so p must be even too. Thus we can write $p = 2k$ for some integer k, whence substitution and cancellation yield $q^2 = 2k^2$. This however means that q is even too, and contradicts the assumption that the fraction $\frac{p}{q}$ was in lowest terms. □

6. Least Upper Bounds via Decimals

Prior to coming to this course, you probably thought of real numbers as (possibly nonterminating) decimals. The steps to making this precise include

(1) interpreting decimals as infinite series
(2) proving that the series of (1) always converge
(3) proving there is a one-to-one correspondence between real numbers and decimals which do not end in all 9's (we temporarily call the latter *standard*).
(4) describing how decimal expansions can be used to determine when one real number is smaller than another

We will carry out this program in Chapter 3. In Section 7.1 (which can be read now), we will even see that this point of view can be turned around, i.e., introducing decimals in a purely formal way provides one method of constructing the complete ordered field guaranteed by Theorem 1.14.1.

At this point, we only want to use decimals to bolster our intuition concerning order completeness. To do so, we take Items 1–4 for granted. The order promised by (4) is lexicographic, e.g., if $x = .x_1x_2x_3\dots$ and $y = .y_1y_2y_3\dots$ are standard decimal expansions, then $x < y$ if and only if these expansions are not identical, and $x_i < y_i$ at their first point of disagreement.

Now suppose that S is a non-empty subset of the open interval $(0, 1) := \{x \in \mathbb{R} : 0 < x < 1\}$. Since S is bounded, it should have a least upper bound b. In fact, we can "construct" b. Take b_1 to be the largest integer appearing as the first digit in the standard expansion of some member of S. Next take b_2 to be the largest integer appearing as the second standard digit of some member of S whose first standard digit is b_1. Continue inductively, taking b_{n+1} to be largest integer appearing as the $(n + 1)$'st standard digit in the expansion of

$$\{x \in S : \text{ the standard expansion of } x \text{ begins with } .b_1 \dots b_n \}.$$

Then $b = .b_1b_2b_3\dots$ may end in all 9's, but in any case it will represent the least upper bound of S.

Exercises

Unless otherwise stated, a, b, c, d, and x stand for real numbers stand for real numbers throughout these exercises.

Field Axioms

PROBLEM 1.1. Prove Part (3) of Proposition 1.3.

PROBLEM 1.2. Prove Proposition 1.4.

PROBLEM 1.3. Give examples to show that the hypothesis $x \neq 0$ cannot be omitted from Parts (1), (2) and (4) of Proposition 1.4. Why wasn't this hypothesis included in (3) ?

PROBLEM 1.4. Give another proof of the fact that $0x = 0$ by applying distributivity to the expression $(0 + 1)x$.

PROBLEM 1.5. Prove that $(a + b)(a - b) = a^2 - b^2$.

PROBLEM 1.6. Construct a field with exactly three elements : 0, 1, and α. You need not verify all the axioms, but you should exhibit the addition and multiplication tables.

PROBLEM 1.7. Suppose $a \neq 0$ and $b \neq 0$. Explain why $ab \neq 0$.

PROBLEM 1.8. Write up a solution of the equation $3x + 5 = 7$ for a middle school algebra class. Then write a one or two paragraph discussion of the role of the basic field axioms and properties in your solution.

PROBLEM 1.9. Criticize and correct the following solution of the equation $x^2 = 6x$.

"Dividing by x, we obtain $x = 6$, which is thus the unique solution."

PROBLEM 1.10. Write up a solution of the equation $x^2 - 4x - 21 = 0$ for a middle school algebra class which hasn't yet been learned the quadratic formula. Then write a one or two paragraph discussion of the role of the basic field axioms and properties in your solution.

PROBLEM 1.11. Criticize and correct the following solution of the equation $x + \sqrt{x} = 6$.

"Transposing x and squaring both sides of the equation, we obtain $x = 36 - 12x + x^2$. From here, we tanspose and factor to obtain $(x-4)(x-9) = 0$. Thus the orgininal equation has two solutions, $x = 4$ and $x = 9$."

PROBLEM 1.12. While it is always desirable to check solutions to equations, sometimes it is essential. Explain the relevance of the last four problems to this assertion.

PROBLEM 1.13. Prove that if neither b nor c is zero, then $\frac{a}{b} = \frac{ac}{bc}$. Hint: Compute the product of each side of the equation with bc and compare.

PROBLEM 1.14. Suppose $bd \neq 0$. Prove that $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

Order

PROBLEM 1.15. Prove Parts (3) and (5) of Proposition 1.9.

PROBLEM 1.16. Prove that if $a > b > 0$ then $a^2 > b^2$.

PROBLEM 1.17. Suppose $a > 0, b > 0$, and $a^2 > b^2$. Prove that $a > b$.

PROBLEM 1.18. Prove that $\frac{2}{3} > \frac{1}{2}$.

PROBLEM 1.19. Write up a solution of the inequality $2 - 5x < 9$ for a middle school algebra class. Then write a one or two paragraph discussion of the role of the basic order axioms and properties in your solution.

PROBLEM 1.20. Find all solutions of the equation $|2x| + |x - 3| = 5$.

PROBLEM 1.21. Prove Proposition 1.13.1.

PROBLEM 1.22. Let a, b, c be members of an ordered field. Define what is meant by the *maximum* of a, b, denoted $\max(a, b)$. Then show that $\max(a, b) \leq c$ if and only if $a \leq c$ and $b \leq c$ both hold. This is used in the proof of Proposition 1.13.2.

PROBLEM 1.23. Prove that $|a| \leq b$ if and only if $-b \leq a \leq b$.

PROBLEM 1.24. Solve each of the following four inequalities:
$|x| < 3$, $|x| < -3$, $|x| > 3$, and $|x| > -3$, expressing your answers in terms of intervals of real numbers.

PROBLEM 1.25. Graph the solution of the inequality $|x - 2| < 4$ on a number line.

PROBLEM 1.26. Apply the triangle inequality to $|a| = |(a - b) + b|$ to show that $|a| - |b| \leq |a - b|$ for all $a, b \in \mathbb{R}$. Then use symmetry to conclude that $||a| - |b|| \leq |a - b|$ for all $a, b \in \mathbb{R}$.

PROBLEM 1.27. Suppose $|x - 5| < 1$. Prove the following:

(1) $|x + 8| < 14$
(2) $|x^2 + 3x - 40| \leq 14|x - 5|$
(3) If $|x - 5| < .001$, then $|x^2 + 3x - 40| < .014$

PROBLEM 1.28. Prove that if $|x - \frac{1}{10}| < \frac{1}{20}$, then $|x| > \frac{1}{20}$.

PROBLEM 1.29. Prove that if $|x - 2| < 0.001$ then $|x^2 - 4| < 0.005$.

Completeness

PROBLEM 1.30. Suppose that b and c are both least upper bounds of a subset S of $\mathbb{R}$. Prove that $b = c$.

PROBLEM 1.31. Suppose b is an upper bound of a set S of real numbers. Prove that b is the least upper bound of S if and only if for each $\varepsilon > 0$, there exists an $a \in S$ with $a > b - \varepsilon$.

PROBLEM 1.32. Define *lower bound* and *greatest lower bound*. Then prove that every non-empty subset S of $\mathbb{R}$ which has a lower bound must have a greatest lower bound. Hint : Consider the set $T = \{-a : a \in S\}$.

PROBLEM 1.33. Give an example of a set of real numbers which has an upper bound but does not have a lower bound.

PROBLEM 1.34. Take S $= \{a \in \mathbb{R} : 0 < a$ and $a^2 < 3\}$. Find several upper bounds for S and then find its least upper bound.

PROBLEM 1.35. Find the greatest lower bound and least upper bound of the set $\{\frac{1}{n} : n \in \mathbb{N}\}$.

PROBLEM 1.36. Find the greatest lower bound and least upper bound of $\{\frac{n-1}{n} : n \in \mathbb{N}\}$.

PROBLEM 1.37. Let F be an ordered field. Prove that F^+ does not have a least upper bound. Explain why this doesn't contradict Theorem 1.19.1.

PROBLEM 1.38. Detail the meaning of the phrase "respects all of the ordered field structures" appearing in the statement of Theorem 1.19.2.

Induction

PROBLEM 1.39. Describe four different inductive subsets of $\mathbb{R}$.

PROBLEM 1.40. How many *bounded* inductive subsets of $\mathbb{R}$ are there?

PROBLEM 1.41. Prove that $\mathbb{N}$ is an inductive set.

PROBLEM 1.42. Use induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for every natural number n.

PROBLEM 1.43. Use induction to prove that $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$ for every natural number n.

PROBLEM 1.44. Use induction to prove that if $a > 1$ then $a^n > 1$ for every natural number n. What can you say if $a < 1$?

PROBLEM 1.45. Use induction to prove that if $|a| < 1$ then $|a^n| < 1$ for every natural number n.

PROBLEM 1.46. Use the inductive definition of Example 1.24 to compute $5!$.

PROBLEM 1.47. Use Example 1.24 to explain why it is reasonable to define $0! = 1$.

PROBLEM 1.48. Use induction to prove that every natural number n is greater than or equal to one, and hence positive.

PROBLEM 1.49. Prove that for each natural number n, either $n = 1$ or $n - 1$ is also a natural number.

PROBLEM 1.50. Use the result of the last problem to prove that every non–empty subset S of $\mathbb{N}$ has a smallest member. This is known as the *well-ordering* property of $\mathbb{N}$.

PROBLEM 1.51. How does the result of the last Problem compare with the result of Problem 1.32 ?

PROBLEM 1.52. Is $\mathbb{N}$ a field? How about $\mathbb{Z}$? How about $\mathbb{Q}$? No proofs are required, but detail which field axioms hold in each case.

PROBLEM 1.53. Prove that there is a rational number between any two reals.

Decimals

PROBLEM 1.54. Apply the technique of Section 1.5 to find decimal expansions for the least upper bounds of the following sets:

(1) $\{\frac{12}{25}, \frac{1}{2}, \frac{9}{100}\}$
(2) $\{x \in \mathbb{R}^+ : x \leq \frac{1}{2}\}$
(3) $\{x \in \mathbb{R}^+ : x < \frac{1}{2}\}$
(4) $\{\frac{37}{100} - 10^{-n} : n \in \mathbb{N}\}$

PROBLEM 1.55. (Suggested by Brandon Samples) Suppose a set F is equipped with two binary operations satisfying all the field axioms *except* that $0 = 1$. Show that F has only one element.

CHAPTER 2

Sequences

1. Introduction

DEFINITION 2.1. A *sequence of real numbers* is a function mapping $\mathbb{N}$ into $\mathbb{R}$.

You are probably accustomed to using letters like f or g to denote functions and x or t for their independent variables. In studying sequences, it is traditional to employ letters k, l, m, n for natural numbers and letters near the beginning of the alphabet for the names of the functions themselves. This should not cause any particular problem. If a is a sequence, it is conventional to write a_n in place of $a(n)$; in prose, this is referred to as the *n'th term* of the sequence. We will also write $(a_n)_{n=1}^{\infty}$ or even (a_n) for the function a. The use of parentheses instead of the more traditional braces emphasizes the distinction between functions and their ranges. For example, the sequences $((-1)^n)$ and $\left((-1)^{n+1}\right)$ are different even though they share the range $\{-1, 1\}$. It should also be kept in mind, here and in the sequel, that ∞ does not stand for any real number, but is used as a mnemonic device for an ongoing process.

The usual way to present a sequence is to give a formula for its general term. This was done implicitly in the preceding paragraph. A less precise but often suggestive method is to list the few terms of a sequence, leaving it to the reader to find "the" formula for the general term.

Example 1.24 illustrated the fact that sequences can also be defined inductively (also called *recursively*). The next example reviews Newton's method, an important instance of this procedure you may have seen in calculus.

EXAMPLE 2.2. Fix a real–valued function f defined on some domain $D \subset \mathbb{R}$. Start with any $a_1 \in D$. Assuming a_n has been defined, take a_{n+1} to be the x–coordinate of the point where the line tangent to the graph of f at $(a, f(a))$ intersects the x-axis. The explicit formula is

$$a_{n+1} = a_n - \frac{f(a_n)}{f'(a_n)}. \tag{2.1}$$

The following special case of Example 2.2 will be used to illustrate several concepts in this chapter.

EXAMPLE 2.3. Apply Newton's method with $f(x) = x^2 - 2$ and $a_1 = 2$. The inductive formula can be written as

$$a_{n+1} = \frac{a_n}{2} + \frac{1}{a_n} \text{ for } n \in \mathbb{N}. \tag{2.2}$$

2. Limits

We want to capture the notion of what it means for the terms in a sequence (a_n) to approach a fixed number L. The intuitive idea is that when n is large, the corresponding term a_n should be "close to" L. The difference between two numbers measures how close they are; since we don't really care which is larger, it is more appropriate to use the absolute value of the difference. This leads to the pseudo-definition "When n is large, $|a_n - L|$ is small". But how small is "small" ? The following formal definition says is effect, "You tell me what you want, and I'll arrange it for you".

DEFINITION 2.4. The sequence (a_n) is said to *converge* to the number L if for each number $\epsilon > 0$ there is a natural number N such that $|a_n - L| < \epsilon$ whenever $n \geq N$.

If you haven't seen it before, Definition 2.4 will probably take some getting used to. We begin with an informal investigation.

EXAMPLE 2.5. Consider the sequence given by $a_n = \frac{3n}{3n-10000}$. The first few terms are close to zero, e.g. $a_1 = \frac{-3}{9997}, a_2 = \frac{-6}{9994}$. Looking ahead a bit though, it seems that the numerator might overwhelm the denominator, e.g., $a_{3332} = -2499, a_{3333} = -9999$. Taking a really long term view however, we see that eventually the 10000 in the denominator becomes less relevant and a_n looks like $\frac{3n}{3n} = 1$, e.g., $a_{10000} = \frac{30000}{20000} = 1.5, a_{20000} = \frac{60000}{50000} = 1.2$. The definition is only concerned with the long-term view, so this sequence should converge to 1.

One challenge given to us by the Definition is to guarantee that a_n is within $\frac{1}{10}$ of 1 for sufficiently large n. We will meet this challenge by showing

$$\left| \frac{3n}{3n - 10000} - 1 \right| < \frac{1}{10} \quad \text{whenever } n > 40000. \tag{2.3}$$

To see that this works, suppose $n > 40000$. Since $3n - 10000$ is then positive, algebraic simplification gives

$$\left| \frac{3n}{3n - 10000} - 1 \right| = \left| \frac{10000}{3n - 10000} \right| = \frac{10000}{3n - 10000}.$$

On the other hand, properties of inequalities tell us that $n > 40000$ implies $3n - 10000 > 110000$ whence taking reciprocals and multiplying by 10000 allows us to continue the last display as

$$\left| \frac{3n}{3n - 10000} - 1 \right| = \frac{10000}{3n - 10000} < \frac{10000}{110000} < \frac{1}{10}$$

and we have successfully met the $\frac{1}{10}$-challenge.

Note that we did not need to solve the original inequality (2.3); in fact that inequality holds for quite a few n's below 40000. Definition 2.4 does however require that we meet *every* ϵ-challenge. We'll postpone that worry till the end of the section.

We proceed with a few formal proofs.

PROPOSITION 2.6. *For any real number c, the constant sequence $(c)_{n=1}^{\infty}$ converges to c.*

PROOF. In the notation of Definition 2.4, our sequence is defined by $a_n = c$ for each n and $L = c$ as well. Let $\epsilon > 0$ be given. Take $N = 1$ (*Any* N would do). Now suppose $n \geq N$. Then $|a_n - L| = |c - c| = 0 < \epsilon$ as desired. $\square$

EXAMPLE 2.7. Show that the sequence given by $a_n = \begin{cases} n, & n < 100, \\ 5, & n \geq 100 \end{cases}$ converges to 5.

PROOF. Given $\epsilon > 0$, take $N = 100$. Then whenever $n \geq N$, it follows that $|a_n - L| = |5 - 5| = 0 < \epsilon$ as desired. $\square$

PROPOSITION 2.8. *The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to* 0.

PROOF. Let $\epsilon > 0$ be given.

[This paragraph is not part of the formal proof, but it illustrates how one thinks ahead to find N. We want to guarantee $|\frac{1}{n} - 0| < \epsilon$. But since n is to be a positive integer, the absolute value is redundent. Thus we really want $\frac{1}{n} < \epsilon$, which is equivalent to $n > \frac{1}{\epsilon}$.]

Returning to the formal proof, apply Proposition 1.25 to get a natural number $N > \frac{1}{\epsilon}$. Then $n \geq N$ forces the desired inequality

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

$\square$

DEFINITION 2.9. A sequence which does not converge to any real number is said to *diverge*. If (a_n) does converge to L, we also say that L is the *limit* of the sequence and write $\lim_{n\to\infty} a_n = L$.

The sequence $(\sqrt{n})$ diverges because its terms increase without bound. The sequence $((-1)^n)$ diverges because its terms jump back and forth between 1 and -1, and thus never approach a fixed limit. On the other hand, having the signs of the terms of a sequence alternate does not automatically preclude convergence.

EXAMPLE 2.10. The sequence $\left(\frac{100(-1)^n}{n}\right)$ converges to 0.

PROOF. For every n we have,

$$|a_n - 0| = \left|\frac{100(-1)^n}{n}\right| = \frac{100}{n}, \tag{2.4}$$

and we can argue as in the preceding proof. Indeed, given $\epsilon > 0$, choose a natural number $N > \frac{100}{\epsilon}$. Then assuming $n \geq N$, we can continue the preceding display as

$$|a_n - 0| = \left|\frac{100(-1)^n}{n}\right| = \frac{100}{n} \leq \frac{100}{N} < \epsilon.$$

□

In fact, it often pays to make a preliminary estimate of the target quantity before attacking Definition 2.4. Here is a slightly fancier example.

EXAMPLE 2.11. Show that $\lim_{n\to\infty} \frac{2n^2+3n}{n^2+5} = 2$

OUTLINE. Begin by working on the target quantity:

$$\left|\frac{2n^2+3n}{n^2+5} - 2\right| = \frac{|3n-10|}{n^2+5} \leq \frac{|3n-10|}{n^2} \leq \frac{3n}{n^2} + \frac{10}{n^2} \leq \frac{13}{n} \tag{2.5}$$

Once you justify each step in this computation, you can introduce ϵ and continue as in the last proof. □

We conclude this section with an elegant writeup of the opening example. A little more algebraic creativity is called for because the sign in the denominator is not so cooperative.

EXAMPLE 2.12. Show that $\lim_{n\to\infty} \frac{3n}{3n-10000} = 1$

PROOF. We begin by noting that $3n - 10000 > n$ when $n > 5000$. Thus we have

$$\left|\frac{3n}{3n-10000} - 1\right| = \left|\frac{10000}{3n-10000}\right| < \frac{10000}{n}, \quad \text{whenever } n > 5000. \tag{2.6}$$

Given $\epsilon > 0$, we choose N to be an integer larger than the *maximum* of 5000 and $\frac{10000}{\epsilon}$. The reason for involving 5000 is to make Inequality 2.6 applicable. Indeed, when $n \geq N$, we can continue the preceding computation to get the desired inequality:

$$\left|\frac{3n}{3n-10000} - 1\right| < \frac{10000}{n} \leq \frac{10000}{N} < \epsilon.$$

□

3. Algebra of Limits

In calculus, you learned the definition of derivative and were probably forced to evaluate a few derivatives directly from the definition. This was necessary to understand the concept, but more efficient computational tools are needed to differentiate functions like $\frac{e^{\sin x}}{x^2+1}$. The situation is analogous for convergence of sequences. While Examples 2.8ff and the related exercises were "for your own good", the time has come to develop better computational tools.

PROPOSITION 2.13. *Suppose the sequence* (a_n) *converges to to a real number* $L > 0$. *Then there is a natural number* N *such that* $a_n > 0$ *for all* $n \geq N$.

PROOF. The difference between this situation and those in the preceding section is that we are *given* the existence of a limit. Thus we don't have to show that something is true for every $\epsilon > 0$; we know and can use that fact. Moreover, there's no requirement to use every ϵ; we can choose which one(s) we find conveneint.

So take $\epsilon = L$. By definition of convergence, there is a natural number N such that $n \geq N$ implies $|a_n - L| < \epsilon$. The latter inequality implies $-L < a_n - L$ which in turn implies $a_n > 0$. □

DEFINITION 2.14. A sequence (a_n) is

bounded above if the **set** $\{a_n : n \in \mathbb{N}\}$ has an upper bound;
bounded below if the **set** $\{a_n : n \in \mathbb{N}\}$ has an lower bound;
bounded if it is simultaneously bounded above and below.

PROPOSITION 2.15. *Every convergent sequence is bounded.*

PROOF. Suppose (a_n) converges to L. Taking $\epsilon = 1$ (any *fixed* ϵ would do), we find a natural number N such that $n \geq N$ implies $|a_n - L| < 1$. This means $-1 < a_n - L < 1$. Thus $L - 1 < a_n < L + 1$ for all $n \geq N$. It follows that $\max\{a_1, a_2, \ldots, a_{N-1}, L + 1\}$ provides an upper bound for the sequence. A similar expression provides a lower bound for the sequence. □

As real-valued functions, sequences can be added, subtracted, and multiplied. Division can present a problem since we can't divide by zero. In order to minimize the inconvenience, we agree to broaden the definition of sequence to allow functions whose domains take the form $\{n \in \mathbb{N} : n \geq N\}$ for some fixed natural number N.

EXAMPLE 2.16. For each natural number n take $a_n = 1, b_n = n - 2$, and $c_n = \sin \frac{n\pi}{2}$. We consider $\frac{a}{b}$ to be a sequence with domain $\{n \in \mathbb{N} : n \geq 3\}$; the usual notations for this sequence are $\left(\frac{1}{n-2}\right)_{n=3}^{\infty}$ and $\left(\frac{1}{n-2}\right)_{n\geq 3}$. On the other hand, since $c_n = 0$ for all even n, we do not consider $\frac{a}{c}$ as a sequence at all.

PROPOSITION 2.17. *Suppose* $\lim a_n = L$.

(1) $\lim K a_n = KL$ *for each real number* K.
(2) *If* $L = 0$, *then* $\lim a_n b_n = 0$ *for every bounded sequence* (b_n).

PROOF. (1) Here we are given one limit and must prove another. The strategy in such situations is *not* to start with the given limit, but rather to relate the target quantity for the unknown limit to the target quantity for the known limit. In the present case we write

$$|Ka_n - KL| = |K||a_n - L| \tag{2.7}$$

The idea is that we can control the left hand side of this equation by controlling $|a_n - L|$, and this sets us up to use the known limit.

Now let $\epsilon > 0$ be given. Since we know $\lim a_n = L$, we can find a natural number N so that $n \geq N$ implies $|a_n - L| < \frac{\epsilon}{|K|+1}$. (The "+1" is triviality insurance in case $K = 0$.) Thus assuming $n \geq N$, the preceding display can be continued to yield $|Ka_n - KL| = |K||a_n - L| \leq |K|\frac{\epsilon}{|K|+1} < \epsilon$, as desired.

(2) Once again, we begin by relating the target quantities involved.

$$|a_nb_n - 0| = |b_n||a_n - 0| \tag{2.8}$$

Next we apply the boundedness of (b_n) to fix a positive number M such that $|b_n| < M$ for all n. This allows us to continue the preceding display as

$$|a_nb_n - 0| = |b_n||a_n - 0| \leq M|a_n - 0| \tag{2.9}$$

The rest of the argument is the same as for Part (1). Given $\epsilon > 0$, choose N so that $n \geq N \implies |a_n - 0| < \frac{\epsilon}{M}$. Thus $n \geq N \implies |a_nb_n| < M\frac{\epsilon}{M} = \epsilon$ as required. □

Example 2.18. Suppose (c_n) is a sequence of positive numbers which converges to 1. Prove that the reciprocal sequence $\left(\frac{1}{c_n}\right)$ converges to 1 as well.

Proof. Note first that $\left|\frac{1}{c_n} - 1\right| = \left|\frac{c_n-1}{c_n}\right|$. Choose N_1 such that $n \geq N_1$ implies $|c_n - 1| < \frac{1}{2}$ and note that this in turn implies $c_n > \frac{1}{2}$. Next, choose N_2 such that $n \geq N_2$ implies $|c_n - 1| < \frac{\epsilon}{2}$. Now set $N = \max\{N_1, N_2\}$. Then, whenever $n \geq N$, we have

$$\left|\frac{1}{c_n} - 1\right| = \left|\frac{c_n - 1}{c_n}\right| < 2|c_n - 1| < \epsilon,$$

as desired. □

Proposition 2.19. *Suppose* (a_n) *and* (b_n) *are convergent sequences. Then*

1. $\lim(a_n + b_n) = \lim a_n + \lim b_n$*;*
2. $\lim a_nb_n = \lim a_n \lim b_n$*;*
3. *If* $\lim b_n \neq 0$ *then* $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$.

Proof. Write $\lim a_n = L$ and $\lim b_n = M$.

For (1), given $\epsilon > 0$, choose N large enough so that $n \geq N$ implies both $|a_n - L| < \frac{\epsilon}{2}$ and $|b_n - M| < \frac{\epsilon}{2}$. Then, whenever $n \geq N$, the triangle inequality yields

$$|(a_n+b_n)-(L+M)| = |(a_n-L)+(b_n-M)| \leq |(a_n-L|+|b_n-M| < \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon,$$

and (1) is established.

We have now done enough epsilontics to coast on our earlier work. For (2), we write $a_nb_n = (a_n-L)b_n+Lb_n$. By Part (1), we know $\lim(a_n-L) = 0$ and Proposition 2.15 tells us the sequence (b_n) is bounded. Thus Proposition

2.17 tells us that the sequences $((a_n - L)b_n)$ and (Lb_n) converge to 0 and LM respectively. Thus the proof of (2) is completed by another appeal to Part (1).

For (3), note that the sequence $\left(\frac{1}{M}b_n\right)$ converges to 1, whence Example 2.18 yields $\lim \frac{M}{b_n} = 1$ as well. But then we can apply Part (2) to conclude

$$\lim \frac{a_n}{b_n} = \left(\lim \frac{a_n}{M}\right)\left(\lim \frac{M}{b_n}\right) = \frac{L}{M},$$

and Part (3) is established. □

EXAMPLE 2.20. Evaluate $\lim \frac{2n^3}{5n^3-7n+9}$.

SOLUTION. Divide numerator and denominator by n^3 and apply Propositions 2.6, 2.8, and 2.19 to conclude the sequence converges to $\frac{2}{5}$. □

EXAMPLE 2.21. Explain why the sequence $\left(\frac{2n^4}{5n^3-7n+9}\right)$ diverges.

SOLUTION. If this sequence did converge, so would its quotient with the sequence of the last example. This contradicts Proposition 2.15. □

4. Monotone Sequences

DEFINITION 2.22. A sequence (a_n) is

(1) *increasing* if the $a_{n+1} \geq a_n$ for each $n \in \mathbb{N}$;
(2) *decreasing* if the $a_{n+1} \leq a_n$ for each $n \in \mathbb{N}$;
(3) *monotone* if it is either increasing or decreasing.

If the corresponding inequalities are always strict, the sequence is said to be *strictly increasing, strictly decreasing, strictly monotone,* respectively.

Some authors use the terms *"non–decreasing"* and *"non–increasing"* in the first two parts of Definition 2.22. This and related conventions, such as whether 0 is considered a natural number, should be checked when consulting a new text.

THEOREM 2.23. *Every bounded monotone sequence converges to a real number.*

PROOF. Suppose (a_n) is increasing and bounded. Take L to be the least upper bound of the set $\{a_n : n \in \mathbb{N}\}$. Let $\epsilon > 0$ be given. Since $L - \epsilon$ is not an upper bound of this set, there is an $N \in \mathbb{N}$ with $a_N > L - \epsilon$. Applying monotonicity and the definition of upper bound, we conclude that $-\epsilon < a_n - L < 0$ whenever $n \geq N$. Thus $n \geq N \implies |a_n - L| < \epsilon$ and the proof is complete for increasing sequences. The decreasing case is left for an exercise. □

Although the last proof is short and (hopefully) easy to understand, Theorem 2.23 should be considered a deep result because it depends on the completeness of the real number system; bounded monotone sequences of rational numbers do not always converge to rational numbers. While Theorem 2.23 is often thought of a theoretical tool (showing limits exist without finding them), the following simple observation turns it into a computational technique as well; it is a special case of Proposition 2.30 below and implies that throwing away (or inserting or changing) the first few terms in a sequence does not affect its convergence.

LEMMA 2.24. *If the sequence* (a_n) *converges then* $\lim a_n = \lim a_{n+1}$.

EXAMPLE 2.25. The sequence constructed in Example 2.3 can be shown to be decreasing; clearly its terms are non–negative. Writing L for its limit, Lemma 2.24 yields $L = \frac{L^2+2}{2L}$. This means $L = \sqrt{2}$.

PROPOSITION 2.26. *Let* $a \in \mathbb{R}$.

(1) *If* $a = 1$, *then* $\lim a^n = 1$.
(2) *If* $|a| < 1$, *then* $\lim a^n = 0$.
(3) *If* $a = -1$, *then the sequence* (a^n) *diverges.*
(4) *If* $|a| > 1$, *then the sequence* (a^n) *diverges.*

PROOF. (1) is obvious, so we may as well assume $a \neq 1$. Lemma 2.24 tells us that if $\lim a^n := L$ exists at all, then $L = \lim a^{n+1} = aL$. Since $a \neq 1$, this forces $L = 0$. Applying the definition of convergence with $\epsilon = 1$, there would have to exist a natural number N for which $|a^N - 0| < 1$ and that can only happen when $|a| < 1$. This establishes (3) and (4).

When $0 \leq |a| < 1$, an inductive argument tells us the sequence $(|a|^n)$ is decreasing. Since it is also bounded, Theorem 2.23 guarantees that $(|a|^n)$ converges, and the preceding paragraph guarantees that its limit must be 0. The proof of (2) is completed by noting that $|a^n - 0| = ||a|^n - 0|$ for all a and n, so that $\lim |a|^n = 0$ implies $\lim a^n = 0$ as well. □

5. Subsequences

In a nutshell, subsequences are composites of sequences. Your reaction to the topic may mirror your experience with composites of functions in calculus: the chain rule takes a little getting used to, but once understood, it becomes the most powerful differentiation technique. In the interests of clarity, we revert to the usual functional notation in this section, writing $a(n)$ instead of a_n.

DEFINITION 2.27. Suppose a is a sequence and b is a strictly increasing sequence of natural numbers. Then the composite sequence $a \circ b$ is said to be a *subsequence* of a.

EXAMPLE 2.28. Think of the sequence a as a list. The subsequence corresponding to $b(n) = n + 1$ is constructed by knocking off the first entry on the original list and renumbering the rest. Similarly the subsequence built from $b(n) = 2n$ is constructed by alternately striking and keeping entries on the original list.

LEMMA 2.29. *Let b be a strictly increasing sequence of natural numbers. Then $b(n) \geq n$ for each n.*

HINT. Argue inductively. □

PROPOSITION 2.30. *All subsequences of a convergent sequence converge to the same limit.*

PROOF. Suppose $\lim a(n) = L$ and b is a strictly increasing sequence of natural numbers. Given $\epsilon > 0$, find N so that $|a(n) - L| < \epsilon$ whenever $n \geq N$. Thus the lemma tells us $|a(b(n)) - L| < \epsilon$ whenever $n \geq N$ as desired. □

EXAMPLE 2.31. Proposition 2.30 can be used in a negative way. For example, one way to establish the divergence of the sequence $((-1)^n)$ is to note that it has subsequences converging to different values.

PROPOSITION 2.32. *Every sequence has a monotone subsequence.*

PROOF. Let a be a sequence.

Call a natural number n *peak* if $a(m) \leq a(n)$ for all $m \geq n$. We divide the rest of the proof into two cases depending on how many peak numbers there are. In both cases, we will define a strictly increasing sequence b inductively.

Suppose first that there is a largest peak number (or none at all). Take b(1) larger than all the peak numbers. Assuming $b(n)$ has been defined, the non–peakness of $b(n)$ allows us to find a natural number $b(n+1) > b(n)$ with $a(b(n+1)) > a(b(n))$. The resulting subsequence $a \circ b$ is strictly increasing.

It remains to consider the case when there is no largest peak number. In this case, take $b(1)$ to be any peak number and assuming $b(n)$ has been defined, take $b(n+1)$ to be any peak number larger than $b(n)$. The resulting subsequence $a \circ b$ is decreasing. □

THEOREM 2.33. *Every bounded sequence has a subsequence which converges to a real number.*

PROOF. Combine Theorem 2.23 and Proposition 2.32. □

It is time to return to the traditional subscript notation for sequences. The standard usage for a subsequence of $(a_n)_{n=1}^{\infty}$ is $(a_{n_k})_{k=1}^{\infty}$. The subtlety here is that while n stands for a natural number in the original sequence, it actually denotes a strictly increasing *sequence* of natural numbers in the subsequence.

6. Cauchy Sequences

Intuitively, a Cauchy sequence is one whose terms are eventually close to each other.

DEFINITION 2.34. A sequence (a_n) is *Cauchy* if for each $\epsilon > 0$ there is a natural number N such that $|a_m - a_n| < \epsilon$ for all natural numbers m and n greater than or equal to N.

PROPOSITION 2.35. (1) *Every convergent sequence is Cauchy.*
(2) *Every Cauchy sequence is bounded.*
(3) *If a Cauchy sequence has a convergent subsequence, then the whole sequence is convergent.*
(4) *Every Cauchy sequence of real numbers has a real limit.*

PROOF. For (1), use an "$\frac{\epsilon}{2}$" argument based on the triangle inequality:

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \leq |a_m - L| + |a_n - L|.$$

For (2), apply the definition with $\epsilon = 1$.

For (3), we revert to honest function notation. We are given a Cauchy sequence a and a strictly increasing sequence b so that $(a(b(n)))$ converges to some number L. We must show that $\lim a(n) = L$ as well. As usual, our hook is the triangle inequality:

$$|a(n) - L| \leq |a(n) - a(b(n))| + |a(b(n)) - L|.$$

Given $\epsilon > 0$, we apply the two hypotheses to find a natural number N so that $n, m \geq N$ simultaneously imply $|a(n) - a(m)| < \frac{\epsilon}{2}$ and $|a(b(n)) - L| < \frac{\epsilon}{2}$. Now if $n \geq N$, then $b(n) \geq N$ as well and thus the last display yields

$$|a(n) - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

as desired.

For (4), use Proposition 2.34 to get a convergent subsequence, and then apply Part (3). □

Cauchy sequences, like subsequences, can be used in a negative way to establish divergence of a sequence. For example, successive terms in $((-1)^n)$ differ by 2 in absolute value, so this sequence is not Cauchy and hence diverges by Proposition 2.35(1). Positive (and more serious) applications of the Cauchy concept will appear in later chapters.

The main emphasis in this course is on *real–valued* sequences. It should be mentioned however that the *definitions* of this chapter make sense in arbitrary ordered fields. This is not the case for all the *results* of the chapter because we have made use of the least upper bound property. This situation is summarized by the following result which will be explored in the exercises; it will not be needed in the sequel.

PROPOSITION 2.36. *The following are equivalent for an ordered field F.*
(1) *Every non–empty subset of F which has an upper bound has a least upper bound.*
(2) *Every bounded monotone sequence in F is convergent.*
(3) *Every Cauchy sequence in F is convergent.*

7. Applications of Calculus

Up to this point, our study of convergence has been rather rigorous and self–contained. It is possible to use what we've learned so far to build a correspondingly rigorous foundation for calculus. We will do a little of this in Chapter 4, but you will have to wait for MATH 4100 and MATH 4500 for fuller treatments; the former concentrates on a logical development of the concepts involved, while the latter is concerned with "practical" issues like minimizing round-off error and "speeding up" convergence.

For the moment, we will adopt a more relaxed attitude, taking what you've learned in calculus for granted. The key observation is that each function $f : \mathbb{R} \to \mathbb{R}$ implicitly defines a sequence $(f(n))$ so that evaluation of $\lim_{x\to\infty} f(x)$ automatically determines the behavior of the sequence. The following version of L'Hôpital's rule is especially useful.

THEOREM 2.37. *Suppose f and g are differentiable functions satisfying*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \text{ or } \lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

as long as the latter limit exists or is infinite.

EXAMPLE 2.38. Evaluate $\lim_{n\to\infty} \frac{\ln n}{n}$.

SOLUTION. Technically we should go through the following steps.

(1) Introduce functions $f(x) = \ln x$ and $g(x) = x$.
(2) Note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$.
(3) Evaluate $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{1}{x} = 0$.
(4) Apply Theorem 2.37 to conclude $\lim_{x\to\infty} \frac{\ln x}{x} = 0$.
(5) Conclude that the sequence $\left(\frac{\ln n}{n}\right)$ converges to 0.

In practice, it is acceptable to abbreviate the process, merely writing

$$\lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

As you may remember from calculus, it is however critical to check (2)— at least mentally—since failure to do so can lead to incorrect results. $\square$

EXAMPLE 2.39. Evaluate $\lim_{n\to\infty} \sqrt[n]{n}$.

SOLUTION. A complete write–up makes the following points.

(1) Introduce $f(x) = x^{\frac{1}{x}}$
(2) Note that $\ln f(x) = \frac{\ln x}{x}$.
(3) Apply the previous example to conclude that $\lim_{x\to\infty} \ln f(x) = 0$.
(4) Apply the continuity of the inverse of the logarithm function (better known as the exponential function) to conclude that $\lim_{x\to\infty} f(x) = e^0 = 1$.
(5) Conclude that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

For an abbreviated presentation, write

$$\lim_{n\to\infty} \ln \sqrt[n]{n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$
$$\text{so} \qquad \lim_{n\to\infty} \sqrt[n]{n} = e^0 = 1.$$

□

Exercises

In the following exercises, $a = (a_n), b = (b_n)$, and $c = (c_n)$ stand for sequences of real numbers. Formal proofs are not required in problems requesting examples or computations.

Newton's Method

PROBLEM 2.1. Draw a graph illustrating the construction of a_2 from a_1 via Newton's method.

PROBLEM 2.2. Derive Equation 2.1.

PROBLEM 2.3. Derive Equation 2.2.

PROBLEM 2.4. Graph the function $f(x) = x^2 - 2$ of Example 2.3, draw the line tangent to the curve at the point $(2, f(2))$, and label the terms $a_1 = 2$ and $a_2 = \frac{3}{2}$ on the x–axis.

PROBLEM 2.5. Compute the first five terms of the sequence from Example 2.3, and compare their decimal expansions (to five decimal places) with the decimal expansion of $\sqrt{2}$.

PROBLEM 2.6. There are three ways Newton's method can fail to generate a_3: a_2 can lie outside the domain of the function f, or $f'(a_2)$ can be zero, or f can fail to be differentiable at a_2. Illustrate each of these possibilities graphically.

Definition of Limit

PROBLEM 2.7. Prove that $|\frac{5}{n+3}| < \frac{1}{10}$ whenever $n \geq 48$.

PROBLEM 2.8. Find a natural number N so that $|\frac{5}{n+3}| < \frac{1}{100}$ whenever $n \geq N$, and prove that it works.

PROBLEM 2.9. Use Definition 2.4 to prove that the sequence $\left(\frac{10}{n}\right)$ converges to 0.

PROBLEM 2.10. Use Definition 2.4 to prove that the sequence $\left(\frac{n}{n+3}\right)$ converges to 1.

PROBLEM 2.11. Define $a_n = \begin{cases} 500, & n < 1000, \\ \frac{2n}{n+3}, & n \geq 1000 \end{cases}$. Use Definition 2.4 to prove that this sequence converges to 2.

Problem 2.12. Solve the inequality $3n - 5 > n$. Then use Definition 2.4 to prove that the sequence $\left(\frac{7n+5}{3n-5}\right)$ convergess to $\frac{7}{3}$.

Problem 2.13. Prove that the constant sequence (0) does *not* converge to any number other than zero.

Problem 2.14. In each part, find a natural number N so that $n \geq N$ implies $|a_n - L| < 0.001$. Formal proofs are not required, but your choice of N should be economical, e.g. don't take $N = 5$ if $N = 4$ will work.

(1) $a_n = \frac{1}{n}$
(2) $a_n = \frac{3}{n-170}$
(3) $a_n = \frac{n}{n+2}$
(4) $a_n = \frac{n^2-4}{n^2-3n+2}$
(5) $a_n = \frac{10(-1)^n}{n}$

Problem 2.15. Write out formal justifications of your answers to the preceding problem.

Problem 2.16. Use Definition 2.4 to prove that the sequences in Problem 2.14 do indeed converge.

Algebra of Limits

Problem 2.17. Suppose (a_n) converges to the real number L.

(1) Prove that if $L < 0$, then there is some natural number N such that $a_n < 0$ for all $n \geq N$.
(2) Prove that if $a_n \geq 0$ for all n, then $L \geq 0$.
(3) Give an example to show that $a_n > 0$ for all n does not imply $L > 0$.

Problem 2.18. Find an expression for the lower bound in the proof of Proposition 2.15.

Problem 2.19. Find upper and lower bounds for the sequences of Problem 2.14.

Problem 2.20. Give an example of a sequence which is bounded above, but not bounded below.

Problem 2.21. Give an example of a bounded sequence which is divergent.

Problem 2.22. Prove that the sequence (a_n) is bounded if and only if the sequence $(|a_n|)$ is bounded above.

Problem 2.23. Find the domain of the sequence $\frac{a}{b}$ if $a_n = n^2 - 9n + 20$ and $b_n = n^2 - 6n + 8$ for all n.

Problem 2.24. Prove that no sequence can have two different limits. There are many ways to do this including directly from Definition 2.4, by reduction to Problem 2.13, or via Problem 2.17.

PROBLEM 2.25. Give an example of two divergent sequences whose sum is convergent.

PROBLEM 2.26. Prove that if (a_n) converges and (b_n) diverges, then the sequence $(a_n + b_n)$ diverges.

PROBLEM 2.27. Give examples to show that each of the following can happen when $\lim a_n = \lim b_n = 0$.

(1) $\left(\frac{a_n}{b_n}\right)$ converges to 0.
(2) $\left(\frac{a_n}{b_n}\right)$ converges to 17.
(3) $\left(\frac{a_n}{b_n}\right)$ is unbounded.
(4) $\left(\frac{a_n}{b_n}\right)$ is bounded but divergent.
(5) $\left(\frac{a_n}{b_n}\right)$ is not considered a sequence.

PROBLEM 2.28. Use the methods of Examples 2.20 and 2.21 to evaluate the following limits:

(1) $\lim \frac{2n^2-3n+5}{7n^2+4n-12}$.
(2) $\lim \frac{2n+5}{7n^2+4n-12}$.
(3) $\lim \frac{2n^3-3n+5}{7n^2+4n-12}$.

PROBLEM 2.29. Let p and q be polynomials. Generalize Examples 2.20 and 2.21 by determining when the sequence $\left(\frac{p(n)}{q(n)}\right)$ converges.

Monotone Sequences

PROBLEM 2.30. Prove that every increasing sequence is bounded below.

PROBLEM 2.31. Give an example of a sequence which is not monotone.

PROBLEM 2.32. Prove that the sum of two increasing sequences is increasing.

PROBLEM 2.33. Give an example of two monotone sequences whose sum is not monotone.

PROBLEM 2.34. Prove Lemma 2.24.

PROBLEM 2.35. Define a sequence recursively by $a_1 = 3$ and $a_{n+1} = \frac{a_n+7}{2}$. Prove that this sequence converges and find its limit.

PROBLEM 2.36. Prove that the sequence of Example 2.3 converges to $\sqrt{2}$:

(1) Prove by induction that $a_n > 0$ for all n.
(2) Prove by induction that $a_n^2 - 2 > 0$ for all n.
(3) Divide both sides of the last inequality by $2a_n$ to prove that $a_{n+1} \leq a_n$ for all n.

(4) Apply Theorem 2.23 and Lemma 2.24.

Problem 2.37. Use Newton's method (Example 2.2) to construct a sequence of rational numbers which converges to $\sqrt{3}$ and prove that it works.

Subsequences

Problem 2.38. Take $a(n) = \frac{(-1)^n n}{n+5}$. For each of the following sequences b, find a formula for the subsequence $a \circ b$ and list the first five terms of that subsequence.

(1) $b(n) = n$.
(2) $b(n) = n + 1$.
(3) $b(n) = 2n$.
(4) $b(n) = n^2$.
(5) $b(n) = 2^n$.

Problem 2.39. Prove Lemma 2.29.

Problem 2.40. Fill in the details of Example 2.31.

Problem 2.41. Find the peak numbers for the sequence $a_n = \frac{(-1)^n}{n}$ and construct two monotone subsequences of this sequence.

Problem 2.42. Give an example of an unbounded sequence which has a convergent subsequence.

Cauchy Sequences

Problem 2.43. Fill in the details of the proofs of Parts (2) and (3) of Proposition 2.35.

Problem 2.44. Prove that the sum of two Cauchy sequences is again Cauchy.

Problem 2.45. Determine which results from this chapter are valid in the field of rational numbers. No proofs are required, but you should give counterexamples whenever appropriate.

Applications of Calculus

Problem 2.46. Give an example of a misuse of L'Hôpital's rule which leads to an incorrect result.

Problem 2.47. Suppose $a_n \le b_n \le c_n$ for all n and $\lim a_n = \lim c_n = L$. Prove that $\lim b_n = L$ as well. This is called the "squeeze principle". (The general result can be reduced to the case $a_n \equiv 0$.)

Problem 2.48. Prove the following. The argument for Part (1) can be based on Probem 1.26 or on Problem 2.17

(1) If $\lim a_n = L$ then $\lim |a_n| = |L|$.

(2) If $\lim |a_n| = 0$ then $\lim a_n = 0$.
(3) If (a_n) converges to a non-zero number, then $((-1)^n a_n)$ diverges.

PROBLEM 2.49. Evaluate the following limits. You may use anything you know.

(1) $\lim \frac{2n^2-3n+5}{7n^2+4n-12}$
(2) $\lim (1.1)^n$
(3) $\lim (0.9)^n$
(4) $\lim \frac{(\ln n)^2}{n}$
(5) $\lim \frac{e^n}{n^2}$
(6) $\lim \frac{n^2+1}{n \ln n}$
(7) $\lim \frac{\sqrt{n^2+1}}{n}$
(8) $\lim \frac{2^n}{3^n}$
(9) $\lim \frac{\ln(\ln n)}{n \ln n}$
(10) $\lim \frac{\arctan 2n}{\arctan 3n}$
(11) $\lim (\sqrt{n+1} - \sqrt{n})$
(12) $\lim (\sqrt{n^2+n} - \sqrt{n^2-n})$
(13) $\lim (1+\frac{1}{n})^n$
(14) $\lim (\ln n)^{\frac{1}{n}}$
(15) $\lim \frac{\sin n}{n}$
(16) $\lim n \sin \frac{1}{n}$
(17) $\lim \frac{(-1)^n \sin n}{n}$
(18) $\lim (-1)^n n \sin \frac{1}{n}$

CHAPTER 3

Series

It is occasionally useful to allow sequences to have negative indices. Accordingly, we extend the definition of sequence to include any real–valued function whose domain takes the form $\{n \in \mathbb{Z} : n \geq N\}$ for some *integer* N.

1. Introduction

DEFINITION 3.1. Let $(a_n)_{n\geq N}$ be a sequence of real numbers. Define a new sequence $(s_n)_{n\geq N}$ inductively by setting $s_N = a_N$ and $s_{n+1} = s_n + a_{n+1}$ for $n \geq N$. We call s_n the *n'th partial sum of the original sequence*; it is also denoted by $\sum_{i=N}^{n} a_i$.

EXAMPLE 3.2.

(1) Take $a_n = n$. The first three partial sums of this sequence are given by $s_1 = 1, s_2 = 1+2 = 3$, and $s_3 = 1+2+3 = 6$. In general, $s_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$ as we saw in Exercise 1.42.

(2) Take $a_n = \frac{1}{n} - \frac{1}{n+1}$. Then $s_1 = 1 - \frac{1}{2} = \frac{1}{2}$,

$$s_2 = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) = 1 - \frac{1}{3} = \frac{2}{3}, \text{ and in general}$$

$$s_n = (1-\frac{1}{2}) + (\frac{1}{2}-\frac{1}{3}) + \cdots + (\frac{1}{n}-\frac{1}{n+1}) = 1 - \frac{1}{n+1} = \frac{n}{n+1}.$$

Note that the telescoping nature of these computations would have been less clear if we had "simplified" the original formula for a_n. In fact, if we had been given $a_n = \frac{1}{n(n+1)}$, we would have been better off using partial fractions to "discover" the original formula.

(3) Take $a_n = \frac{1}{n(n-1)}$. The domain of this sequence is $n \geq 2$. Thus s_1 is not defined for this sequence. We have $s_2 = \frac{1}{2}$, $s_3 = \frac{1}{2} + \frac{1}{6} = \frac{2}{3}$, and $s_4 = \frac{2}{3} + \frac{1}{12} = \frac{3}{4}$. An inductive argument can be used to show that $s_n = \frac{n-1}{n}$ for each $n \geq 2$.

DEFINITION 3.3. A sequence $(a_n)_{n\geq N}$ is said to be *summable* if the corresponding sequence $(s_n)_{n\geq N}$ of partial sums is convergent. In this case, we write $\sum_{i=N}^{\infty} a_i$ for $\lim_{n\to\infty} s_n$; this number is called the *sum* of the original sequence $(a_n)_{n\geq N}$.

EXAMPLE 3.4. The constant sequence (1) is not summable since the corresponding sequence $(n)_{n=1}^{\infty}$ of partial sums does not have a (finite) limit.

On the other hand, the sequence $\left(\frac{1}{n(n+1)}\right)$ of Example 3.2(2) is summable and we have $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{n\to\infty} \frac{n}{n+1} = 1$.

It should be pointed out that the letters used for indices in sequences are "dummy variables" and can be changed at will. Thus $\left(\frac{1}{n(n-1)}\right)_{n\geq 2}$ and $\left(\frac{1}{i(i-1)}\right)_{i\geq 2}$ denote the same sequence, both $\sum_{n=2}^{10} \frac{1}{n(n-1)}$ and $\sum_{i=2}^{10} \frac{1}{i(i-1)}$ stand for the tenth partial sum of this sequence, and both $\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ and $\sum_{i=2}^{\infty} \frac{1}{i(i-1)}$ denote the sum of this sequence. Similarly, i can be replaced by any letter *except* n in the expression $\sum_{i=2}^{n} \frac{1}{i(i-1)}$.

PROPOSITION 3.5. *If the sequence* (a_n) *is summable, then* $\lim a_n = 0$.

PROOF. Write S for the sum of the sequence. Applying Lemma 2.24, we have $\lim s_n = \lim s_{n-1} = S$. Since $a_n = s_n - s_{n-1}$ for all sufficiently large n, the algebra of limits yields $\lim a_n = S - S = 0$. □

Proposition 3.5 is a powerful tool for showing sequences are *not* summable. Its converse however is false; Parts 4) and 5) of the next example show that Proposition 3.5 it can *never* be used to show a sequence is summable.

EXAMPLE 3.6.
(1) The sequence $\left(\frac{n}{n+1}\right)$ is not summable since its terms approach a number other than zero.
(2) The sequence $((-1)^n)$ is not summable because its terms do not approach any number at all.
(3) Proposition 3.5 *does not help us decide* whether $\left(\frac{1}{n(n+1)}\right)$ is summable; we happen to know that this sequence is summable because of our work in Example 3.4.
(4) Proposition 3.5 *does not help us decide* whether the sequence $\left(\frac{1}{n}\right)$ is summable; we will soon see that this sequence is not summable.
(5) Proposition 3.5 *does not help us decide* whether $\left(\sqrt{n} - \sqrt{n-1}\right)$ summable. Although we have $\lim \sqrt{n} - \sqrt{n-1} = \lim \frac{1}{\sqrt{n}+\sqrt{n-1}} = 0$, the n'th partial sum of this sequence telescopes to $\sqrt{n}$ and hence this sequence is not summable.

The last part of the preceding example illustrates a sneaky method for constructing pathological examples: *start* with a formula for partial sums and then *subtract* to get the "original" sequence. In the last example, this means starting with the formula $s_n = \sqrt{n}$, whence $a_1 = s_1 = 1$ while $a_n = s_n - s_{n-1} = \sqrt{n} - \sqrt{n-1}$ for $n > 1$.

DEFINITION 3.7. An *(infinite) series* is a pair of sequences $(a_n)_{n\geq N}$ and $(s_n)_{n\geq N}$ where the latter sequence is the sequence of partial sums of the former sequence. It is denoted by $\sum_{n=N}^{\infty} a_n$. We call a_n the n'th *term* of the series and we call s_n the n'th *partial sum* of the series.

DEFINITION 3.8. The series $\sum_{n=N}^{\infty} a_n$ *converges* if the sequence $(a_n)_{n \geq N}$ is summable. In this case, the sum S of the sequence $(a_n)_{n \geq N}$ is referred to as the *sum* of the series and we say that the series *converges to* S; we also write $S = \sum_{n=N}^{\infty} a_n$. A series which does not converge is said to *diverge*.

This is the traditional terminology. The most important thing to keep in mind is that convergence of a series refers to convergence of its sequence of *partial sums*. The notation $\sum_{n=N}^{\infty} a_n$ is ambiguous; one must consider the context to decide whether it stands for a series (i.e., a pair of sequences) or a number (i.e., the sum of a convergent series). We recast our earlier results in the new terminology.

PROPOSITION 3.9 (n'th Term Test). *In order for a series to converge, it is necessary that its terms approach zero.*

PROOF. This is Proposition 3.5. □

EXAMPLE 3.10. (1) The series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ converges to 1.
(2) The series $\sum (-1)^n$ diverges.
(3) The series $\sum \sqrt{n} - \sqrt{n-1}$ diverges.

PROOF. See Example 3.6. □

DEFINITION 3.11. A series of the form $\sum_{n=0}^{\infty} a^n$ where a is a fixed real number is called a *geometric* series.

PROPOSITION 3.12. *The geometric series* $\sum_{n=0}^{\infty} a^n$ *converges to* $\frac{1}{1-a}$ *if* $|a| < 1$. *If* $|a| \geq 1$, *the series diverges.*

PROOF. We may as well assume $a \neq 1$. To derive a general formula for s_n, begin by writing

$$s_n = \sum_{i=0}^{n} a^i = 1 + a + a^2 + \cdots + a^n.$$

Next multiply both sides of this equation by a to obtain

$$as_n = \sum_{i=0}^{n} a^{i+1} = a + a^2 + a^3 + \cdots + a^{n+1}.$$

Taking the difference between the last two equations yields $(1-a)s_n = 1 - a^{n+1}$. Dividing by $1-a$ yields the desired result $s_n = \frac{1-a^{n+1}}{1-a}$ for each $n \geq 0$; this formula can also be established by induction. The algebra of limits shows that the sequence (s_n) converges to $\frac{1}{1-a}$ if $|a| < 1$ but diverges otherwise. □

PROPOSITION 3.13. *Suppose the series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge to* A *and* B *respectively and* K *is a real number.*

(1) *The series* $\sum_{n=1}^{\infty}(a_n + b_n)$ *converges to* $A+B$.
(2) *The series* $\sum_{n=1}^{\infty} Ka_n$ *converges to* KA.

Proof. Write (r_n), (s_n), and (t_n) for the sequences of partial sums of the series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$, and $\sum_{n=1}^{\infty}(a_n + b_n)$ respectively. Then $t_n = r_n + s_n$ for each n whence $\lim t_n = A + B$ by the algebra of limits. This establishes (1); the proof of (2) is similar. □

Proposition 3.14. *Omitting or changing the first few terms of a series does not affect its convergence.*

Proof. To see that omitting the first N terms of $\sum_{n=1}^{\infty} a_n$ does not affect convergence, set $b_n = a_{n+N}$ and write s_n and t_n for the n'th partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Then $t_n = s_{n+N} - s_N$ for each n, whence convergence of the partial sum sequence (s_n) implies convergence of the sequence (t_n) and vice versa.

Changing the first few terms in a sequence can be done in two stages — omit and insert. By the preceding paragraph, neither of these processes affects convergence, so this observation completes the proof. □

2. Comparison

As you might imagine, it can be difficult to evaluate sums of series. The next few results develop Theorem 2.23 into a tool for testing convergence of many series whose sums can not be explicitly evaluated.

Proposition 3.15. *A series with non–negative terms is convergent if and only if its sequence of partial sums is bounded.*

Proof. The non–negativity of the terms means the corresponding sequence (s_n) of partial sums is increasing. Proposition 2.15 and Theorem 2.23 tell us that this sequence of partial sums converges if and only if it is bounded. □

Example 3.16. The *harmonic* series $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent.

Proof. An inductive argument shows that the corresponding sequence (s_n) satisfies $s_{2^n} \geq \frac{n+2}{2}$ for each n. The unboundedness of the latter sequence shows that the series diverges. □

Alternate proof. Since the function $f(x) = \frac{1}{x}$ is non–increasing, we have

$$\frac{1}{n} \leq \int_{n-1}^{n} \frac{1}{x} dx = \log n - \log(n-1)$$

for each $n \geq 2$. Adding these inequalities inductively, we find that $s_n \geq \log n$ for each $n \in \mathbb{N}$. This again yields the unboundedness of the sequence (s_n) of partial sums and hence the divergence of the series $\sum \frac{1}{n}$. □

Proposition 3.17 (Comparison Test). *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with $0 \leq a_n \leq b_n$ for all sufficiently large n.*

(1) *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also converges.*
(2) *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ also diverges.*

PROOF. Since changing finitely many terms in a series does not affect its convergence, we may as well assume $0 \leq a_n \leq b_n$ for *all* n. Write (s_n) and (t_n) for the sequences of partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. An inductive argument shows that $s_n \leq t_n$ for each n. For (1), argue that convergence of $\sum_{n=1}^{\infty} b_n$ implies boundedness of the sequence (t_n), which implies boundedness of the sequence (s_n), which in turn yields convergence of the series $\sum_{n=1}^{\infty} a_n$. (2) is the contrapositive of (1). □

EXAMPLE 3.18. The series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

PROOF. For each n, take $a_n = \frac{1}{n^2}$ and $b_n = \frac{2}{n(n+1)}$. We have $0 \leq a_n \leq b_n$ for each n, and $\sum_{n=1}^{\infty} b_n$ converges by Example 3.4. Thus $\sum_{n=1}^{\infty} a_n$ converges by comparison. □

There are two subtleties involved in applying the comparison test. First, one must choose an appropriate auxiliary series, and then one must prove a string of inequalities. The following variation of the test often simplifies the process.

PROPOSITION 3.19 (Ratio-Comparison Test). *Suppose* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are series with non–negative terms.*

(1) *If* $\sum_{n=1}^{\infty} b_n$ *converges, and the sequence* $\left(\frac{a_n}{b_n}\right)$ *has a finite limit, then the series* $\sum_{n=1}^{\infty} a_n$ *also converges.*
(2) *If* $\sum_{n=1}^{\infty} b_n$ *diverges, and the sequence* $\left(\frac{a_n}{b_n}\right)$ *either has a non–zero limit or diverges to* ∞*, then the series* $\sum_{n=1}^{\infty} a_n$ *also diverges.*
(3) *If* $\lim \frac{a_n}{b_n}$ *is a non-zero finite number, then the two series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *either both converge or both diverge.*

PROOF. For (1), suppose $\lim \frac{a_n}{b_n} = L$. For all sufficiently large n, we have $\frac{a_n}{b_n} \leq L + 1$ whence $0 \leq a_n \leq (L + 1)b_n$. The comparison test thus yields the convergence of $\sum_{n=1}^{\infty} a_n$.

(2) and (3) can be established by similar arguments or by applying (1) with the roles of a_n and b_n reversed. □

Part (3) of the previous proposition is easiest to apply since you can just "take the limit and ask questions afterwards". As for the choice of the auxiliary series $\sum_{n=1}^{\infty} b_n$, we basically have three choices at this point:

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$ (harmonic series, diverges),
(2) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ (Example 3.18, converges), and
(3) $\sum_{n=1}^{\infty} c^n$ (geometric series, converges for $0 \leq c < 1$ and diverges for $c \geq 1$).

EXAMPLE 3.20. Determine which of the following series $\sum_{n=1}^{\infty} a_n$ converge.

(1) $a_n = \frac{1}{3n+5}$,

(2) $a_n = \frac{n}{3n+5}$,
(3) $a_n = \frac{n^2-3n+5}{n^4+7n+2}$
(4) $a_n = \frac{n^2-3n+5}{n^5+7n+2}$
(5) $a_n = \frac{1}{\ln(n+2)}$,
(6) $a_n = \frac{n}{4^n}$.

SOLUTION. (1) Take $b_n = \frac{1}{n}$. Then $\lim \frac{a_n}{b_n} = \frac{1}{3}$ so $\sum a_n$ diverges.

(2) You could use the same comparison as in (1) but it is easier to obtain divergence via the n'th term test. It is always a good idea to check the n'th term test before getting involved with any fancier tests.
(3) Obtain convergence by comparing with $\sum \frac{1}{n^2}$.
(4) Obtain convergence by comparing with $\sum \frac{1}{n^2}$.
(5) Obtain divergence by comparing with $\sum \frac{1}{n}$.
(6) Obtain convergence by comparing with $\sum \frac{1}{2^n}$. Indeed,

$$\lim \frac{a_n}{b_n} = \lim \frac{n2^n}{4^n} = \lim \frac{n}{2^n} = \lim \frac{1}{(\ln 2)(2^n)} = 0$$

by l'Hôpital's rule. Note how sneaky this is: comparison with $\sum \frac{1}{4^n}$ goes the wrong way to give any useful information, so we use the more delicate comparison to overwhelm the offending factor of n.

□

You should do Exercise 17 now, before learning more convergence tests; this will provide practice with the ratio-comparison test in a distraction–free setting.

3. Justification of Decimal Expansions

We take a short break in our development of convergence tests to fulfill the promise made in Section 1.5.

DEFINITION 3.21. $0, 1, 2, 3, 4, 5, 6, 7, 8,$ and 9 are referred to as *digits.*

(1) A *decimal* is an infinite series $\sum_{n=N}^{\infty} d_n 10^{-n}$ where $N \leq 1$ is an integer, and $(d_n)_{n \geq N}$ is a sequence of digits. We denote this series by $d_N \cdots d_0.d_1 d_2 \cdots$.
(2) As usual, we write $d = d_N \cdots d_0.d_1 d_2 \cdots$ to signify that the series of (1) converges to the real number d. In such a case, we call $d_N \cdots d_0.d_1 d_2 \cdots$ a *decimal expansion* of d and we also call $-d_N \cdots d_0.d_1 d_2 \cdots := \sum_{n=N}^{\infty} -d_n 10^{-n}$ a *decimal expansion* of $-d$.

PROPOSITION 3.22. *Every decimal converges to a real number.*

PROOF. Let $(d_n)_{n \geq N}$ be a sequence of digits. For each $n \in \mathbb{N}$, we have $0 \leq d_n 10^{-n} \leq 9(10^{-n})$. Now $\sum_{n=N}^{\infty} 9(10^{-n})$ is a convergent geometric series, whence $\sum_{n=N}^{\infty} d_n 10^{-n}$ converges by the comparison test. □

The following simple observation makes for more readable proofs.

LEMMA 3.23. *Shifting the decimal point k places to the right multiplies the number to which a decimal converges by a factor of* 10^k.

PROOF. This is a special case of Proposition 3.13.2. □

PROPOSITION 3.24. *Every real number has a decimal expansion.*

PROOF. We use the notation $\lfloor y \rfloor$ to denote the greatest integer in the real number y.

Fix a real number $x \geq 0$. In view of the preceding lemma, we may as well assume $0 < x < 1$. Set $d_0 = s_0 = 0$. Proceeding recursively, define $d_{n+1} = \lfloor 10^{n+1}(x - s_n) \rfloor$ and $s_{n+1} = s_n + d_{n+1}10^{-(n+1)}$. An inductive argument shows that $0 \leq x - s_n < 10^{-n}$ for each n, whence the series $\sum_{n=N}^{\infty} d_n 10^{-n}$ does indeed converge to x. □

PROPOSITION 3.25. *Let $x_N \cdots x_0.x_1x_2x_3 \cdots$ and $y_N \cdots y_0.y_1y_2y_3 \cdots$ be decimal expansions of non-negative real numbers x, y respectively. Suppose there is an integer K such that $x_K < y_K$, while $x_i = y_i$ for all $i < K$. Then $x \leq y$. In order that $x = y$ it is necessary and sufficient that*

(1) $y_K = 1 + x_K$,
(2) $x_i = 9$ *for all* $i > K$, *and*
(3) $y_i = 0$ *for all* $i > K$.

PROOF. In view of Lemma 3.22, we may as well assume $K = 0$. There is also no loss of generality in assuming $\lfloor x \rfloor = 0$, in which case we have

$$x = \sum_{i=1}^{\infty} \frac{x_i}{10^i} \leq \sum_{i=1}^{\infty} \frac{9}{10^i} = 1 \leq \lfloor y \rfloor + \sum_{i=1}^{\infty} \frac{0}{10^i} \leq \lfloor y \rfloor + \sum_{i=1}^{\infty} \frac{y_i}{10^i} = y.$$

Thus $x = y$ precisely when equality holds throughout this display. □

COROLLARY 3.26. *Every real number has a unique decimal expansion which does not end in all 9's.*

This completes the program outlined in Section 1.5. The proof of one additional fact will be left for the exercises.

DEFINITION 3.27. A decimal representation $d_N \cdots d_0.d_1d_2 \cdots$ is *terminating* if $d_i = 0$ for all sufficiently large $i \in \mathbb{N}$; it is *repeating* if there is a $p \in \mathbb{N}$ satisfying $d_{i+p} = d_i$ for all sufficiently large $i \in \mathbb{N}$.

PROPOSITION 3.28. *All decimal representations of rational numbers are repeating and every real number which has a repeating decimal representation must be rational.*

4. Ratio Test

We return to our general discussion of convergence tests. The next test is especially convenient because no auxiliary series is needed to apply it.

PROPOSITION 3.29 (Ratio Test). *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series with positive terms and* $\lim \frac{a_{n+1}}{a_n} = L$ *exists as a finite number or is infinite.*

(1) *If* $L < 1$*, then the series converges.*
(2) *If* $L > 1$*, then the series diverges.*

PROOF. (1) Fix a real number r strictly between L and 1. Since we know $\lim \frac{a_{n+1}}{a_n} < r$, we have $a_{n+1} \leq r a_n$ for all sufficiently large n. Since omitting the first few terms of a series does not affect convergence, we may as well assume $a_{n+1} \leq r a_n$ for *all* n. But then an induction argument shows yields $a_n \leq a_1 r^{n-1}$ for each n and hence $\sum a_n$ converges by comparison with the geometric series $\sum a_1 r^{n-1}$.

While (2) also follows by comparison with an appropriate geometric series, it is easier to note that $a_{n+1} > a_n$ for sufficiently large n whence divergence follows by the n'th term test. □

EXAMPLE 3.30. (1) The ratio test shows that $\sum \frac{n}{4^n}$ converges because $\lim \frac{a_{n+1}}{a_n} = \lim \frac{n+1}{4n} = \frac{1}{4}$. Note that this approach is easier than that of Example 3.20(6). It is also useful to note that the n'th term test can be used in a *positive* way here to conclude that $\lim \frac{n}{4^n} = 0$. Sometimes the easiest way to show that a sequence converges to zero is to show that it is summable!

(2) The ratio test tells us that $\sum \frac{10^n}{n!}$ converges because of the computation $\lim \frac{a_{n+1}}{a_n} = \lim \frac{10}{n+1} = 0$.
(3) The ratio test tells us that $\sum \frac{2^n}{n}$ diverges.
(4) The ratio test *gives no information* concerning either of the series $\sum \frac{1}{n}$ or $\sum \frac{1}{n^2}$ since $\lim \frac{a_{n+1}}{a_n} = 1$ in both cases.
(5)
$$\text{Set } a_n = \begin{cases} \frac{1}{2}, & \text{for } n \text{ odd} \\ \frac{1}{3}, & \text{for } n \text{ even.} \end{cases}$$
$$\text{Then } \frac{a_{n+1}}{a_n} = \begin{cases} \frac{3}{2}, & \text{for } n \text{ odd} \\ \frac{2}{3}, & \text{for } n \text{ even} \end{cases}$$
so $\lim \frac{a_{n+1}}{a_n}$ does not exist. Thus the ratio test *gives no information* concerning $\sum a_n$; it is easily seen to diverge, however, by the n'th term test.

5. Integral Test

So far, our applications of the comparison tests have been limited by our poor stock of auxiliary series. For example, comparisons between $\sum \frac{1}{n\sqrt{n}}$

with the known series $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ both go the wrong way to give any useful information. The integral test remedies this situation.

Suppose f is a continuous function mapping $\mathbb{R}^+$ into itself. Then the definite integral $F(b) = \int_1^b f(x)dx$ exists for each positive number b. In fact, F is a non–decreasing function of b. In view of the least upper bound axiom, the limit $\lim_{b\to\infty} F(b)$ exists if and only if F is bounded. If this is the case, we set $\int_1^\infty f = \lim_{b\to\infty} F(b)$ and say that the *improper integral* $\int_1^\infty f$ *converges*; otherwise, we write $\int_1^\infty f = \infty$ and say that the *improper integral* $\int_1^\infty f$ *diverges*. The alternate proof given in Example 3.16 is a special case of the following argument.

PROPOSITION 3.31 (Integral Test). *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous and non–increasing. Then the series $\sum_{n=1}^\infty f(n)$ converges if and only if the improper integral $\int_1^\infty f$ converges.*

PROOF. Write s_n for the n'th partial sum of the series. Because f is non–increasing, we have $f(i+1) \leq \int_i^{i+1} f \leq f(i)$ for each natural number i. Adding these inequalities for $i = 1 \cdots n$, we obtain $s_{n+1} - s_1 \leq \int_1^{n+1} f \leq s_n$. This means boundedness of the sequence of partial sums is equivalent to boundedness of F so the result follows from Theorem 2.23. □

EXAMPLE 3.32.
(1) The function $f(x) = \frac{1}{x}$ maps $\mathbb{R}^+$ to itself and it is non–increasing since its derivative is always negative. We have $\int_1^\infty \frac{1}{x}dx = \lim_{b\to\infty} \ln b = \infty$ so the series $\sum \frac{1}{n}$ diverges.
(2) For any $p > 1$, $\int_1^\infty \frac{1}{x^p}dx = \lim_{b\to\infty} \frac{1}{p-1}[1 - \frac{1}{b^{p-1}}] = \frac{1}{p-1}$ so the series $\sum \frac{1}{n^p}$ converges.
(3) $\int_1^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty} \arctan b - \arctan 1 = \frac{\pi}{4}$ so the series $\sum \frac{1}{n^2+1}$ converges. You could reach the same conclusion by comparing $\sum \frac{1}{n^2+1}$ with $\sum \frac{1}{n^2}$.
(4) The best way to establish convergence of a series like $\sum \frac{1}{n^3+n^2+2n+2}$ is by comparison with $\sum \frac{1}{n^3}$ — you could use partial fractions to integrate $\frac{1}{x^3+x^2+2x+2}$, but do you really want to?
(5) The integral test cannot help with the series of Example 3.30(5) because it is not monotone.

6. Series with Sign Changes

So far, we have no method of establishing convergence of series having both positive and negative signs. (Of course, the n'th term test can be used to show that some of these series diverge).

DEFINITION 3.33. The series $\sum a_n$ is *absolutely convergent* if the series $\sum |a_n|$ is convergent.

PROPOSITION 3.34 (Absolute Convergence Test). *Each absolutely convergent series is convergent.*

PROOF. Suppose $\sum |a_i|$ converges. Write s_n and t_n for the n'th partial sums of the series $\sum a_i$ and $\sum |a_i|$ respectively. By hypothesis, we know the sequence (t_n) is convergent; in view of Proposition 2.35, it must also be Cauchy. But the triangle inequality shows that $|s_m - s_n| \leq |t_m - t_n|$ for each pair of indices m, n. It follows that the sequence (s_n) is also Cauchy and hence convergent by a second application of Proposition 2.35. □

In particular, this result removes the non–negativity requirement from the ratio test.

COROLLARY 3.35 (Ratio Test). *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series for which* $\lim |\frac{a_{n+1}}{a_n}| = L$ *exists as a finite number or is infinite.*

(1) *If* $L < 1$*, then the series converges.*
(2) *If* $L > 1$*, then the series diverges.*

PROPOSITION 3.36 (Alternating Series Test). *If the sequence* $(b_n)_{n=1}^{\infty}$ *decreases monotonically to zero, then the series* $\sum_{n=1}^{\infty}(-1)^n b_n$ *converges.*

PROOF. Set $a_n = (-1)^n b_n$ and write s_n for the n'th partial sum of the series $\sum_{n=1}^{\infty} a_n$. By monotonicity of the b_n's, we see that $a_{2n-1}+a_{2n}$ is always negative, whence the subsequence $(s_{2n})_{n=1}^{\infty}$ is decreasing. On the other hand, $a_{2n} + a_{2n+1}$ is always positive, so a_1 is a lower bound for the subsequence (s_{2n}). Theorem 2.23 thus tells us that this subsequence converges to some finite limit L. We also have $\lim s_{2n+1} = \lim s_{2n} - b_{2n+1} = L$ so the full sequence (s_n) of partial sums converges to L and the proof is complete. □

DEFINITION 3.37. A convergent series $\sum a_n$ which does not converge absolutely is said to converge *conditionally.*

EXAMPLE 3.38.

(1) The series $\sum \frac{(-1)^n n}{n+1}$ diverges by the n'th term test.
(2) The series $\sum \frac{(-1)^n}{n^2}$ is absolutely convergent.
(3) The series $\sum \frac{(-1)^n}{n}$ is conditionally convergent.

7. Strategy

We have built up quite an arsenal of tests for convergence of series. It is important to organize them for your own use. Start by listing the tests by name and stating each one precisely. (Use the index at the end of this book for this purpose). Next, summarize situations where each is particularly useful. For example, factorials and powers are usually amenable to the ratio test. Finally, write out a plan for dealing with new series, possibly in the form of a flow chart. Strive to try simple tests first. The n'th term test is usually quick to apply mentally. It won't solve all your problems, but you'll be sorry if you overlook it when it can help.

8. Abel's Inequality and Dirichlet's Test

(added by Dave Benson)

In this section, we prove a generalization of the alternating series test. It will be especially handy is Chapter 6.

THEOREM 3.39 (Abel's Inequality). *Suppose that we are given numbers $a_1, \ldots, a_m$ and $b_1, \ldots, b_m$, and suppose that for all n between 1 and $m-1$ we have $b_n \geq b_{n+1} \geq 0$. Set $s_n = a_1 + \cdots + a_n$, and $M = \max\{|s_1|, \ldots, |s_m|\}$. Then*

$$\left|\sum_{i=1}^{m} a_i b_i\right| \leq M b_1.$$

PROOF. We have $a_n = s_n - s_{n-1}$, so

$$\begin{aligned}\sum_{i=1}^{m} a_i b_i &= s_1 b_1 + (s_2 - s_1) b_2 + \cdots + (s_m - s_{m-1}) b_m \\ &= s_1(b_1 - b_2) + s_2(b_2 - b_3) + \cdots + s_{m-1}(b_{m-1} - b_m) + s_m b_m.\end{aligned}$$

Each $|s_n| \leq M$ and $b_n - b_{n+1}$ is positive, so

$$|s_n(b_n - b_{n+1})| \leq M(b_n - b_{n+1}).$$

By the triangle inequality, we have

$$\begin{aligned}\left|\sum_{i=1}^{m} a_i b_i\right| &\leq |s_1(b_1 - b_2)| + \cdots + |s_{m-1}(b_{m-1} - b_m)| + |s_m b_m| \\ &\leq M(b_1 - b_2) + \cdots + M(b_{m-1} - b_m) + M b_m \\ &= M b_1.\end{aligned}$$

□

Terminology. We say that a sequence of positive numbers b_n *tends to zero steadily* if for all $n \geq 1$ $b_n \geq b_{n+1} \geq 0$ and $\lim_{n\to\infty} b_n = 0$.

THEOREM 3.40 (Dirichlet's test for convergence). *Suppose that a_n and b_n are two sequences of numbers, and set $s_n = a_1 + \cdots + a_n$. Suppose that the s_n are bounded, and that the b_n decrease monotonically to zero. Then $\sum_{i=1}^{\infty} a_i b_i$ converges.*

PROOF. We shall check Cauchy convergence of the partial sums of the $a_i b_i$. Given $\varepsilon > 0$, we need to show that there exists $N \in \mathbb{N}$ such that for all $m \geq n \geq N$ we have

$$\left|\sum_{i=n+1}^{m} a_i b_i\right| \leq \varepsilon.$$

In order to apply Abel's test, we must examine the partial sums $\left|\sum_{i=n+1}^{n'} a_i\right|$ for n' between $n+1$ and m. By hypothesis, the partial sums s_n are bounded, say $|s_n| \leq K$ for all $n \geq 0$. By the triangle inequality, we have

$$\left|\sum_{i=n+1}^{n'} a_i\right| = |s_{n'} - s_n| \leq |s_{n'}| + |s_n| \leq 2K.$$

So we need to choose N large enough so that for all $n \geq N$ we have $b_n < \varepsilon/2K$. We can do this since $\lim_{n\to\infty} b_n = 0$. Now we apply Abel's inequality to obtain

$$\left|\sum_{i=n+1}^{m} a_i b_i\right| \leq (2K)b_{n+1} < (2K)(\varepsilon/2K) = \varepsilon.$$

This completes the proof of Cauchy convergence of the partial sums, and so the theorem is proved. □

Exercises

Summability

Problem 3.1. Consider the sequence $\left(\frac{1}{n+3} - \frac{1}{n+4}\right)$. Compute its first five partial sums and guess a general formula for its n'th partial sum.

Problem 3.2. Give an inductive proof of your conjecture from the preceding problem.

Problem 3.3. Prove that the sequence of Problem 1 is summable and find its sum.

Problem 3.4. Prove that the sequence $\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right)$ is summable.

Problem 3.5. Prove that the sequence $\left(\frac{1}{n(n+2)}\right)$ is summable and find its sum. Hint: Guess a general formula for s_n by inspection or use the method of partial fractions to derive it.

Problem 3.6. Find a general formula for the n'th term of the series whose sequence of partial sums is $\left(n^2\right)$.

Problem 3.7. What information does the n'th term test give concerning summability of the following sequences?

(1) $a_n = \arctan n$.
(2) $a_n = \frac{n}{e^n}$.
(3) $a_n = \frac{(-1)^n}{n+1}$.
(4) $a_n = \sin \frac{n\pi}{8}$.

Geometric Series

Problem 3.8. Any series of the form $\sum_{n=N}^{\infty} ca^n$ is considered a geometric series. When does such a series converge and what is its sum when it does converge ?

PROBLEM 3.9. Interpret the repeating decimal $0.999\ldots$ as an infinite series and explain why this number equals 1.

PROBLEM 3.10. Criticise the following "proof" of Proposition 3.12.

Assume $a \neq 1$. Write S for the sum of the series.

$$\text{Then} \quad S = \sum_{i=0}^{\infty} a^i = 1 + a + a^2 + a^3 + \ldots.$$

Next multiply both sides of this equation by a to obtain

$$aS = \sum_{i=0}^{\infty} a^{i+1} = a + a^2 + a^3 + a^4 + \ldots.$$

Taking the difference between the last two equations yields $(1-a)S = 1$. Dividing by $1-a$ yields the desired result $S = \frac{1}{1-a}$ for each $a \neq 1$.

PROBLEM 3.11. Establish the formula for s_n derived in the proof of Proposition 3.12 by induction.

PROBLEM 3.12. Write out the proof of Proposition 3.13(2).

n'th Term and Comparison Tests

PROBLEM 3.13. Give an example of a divergent series whose sequence of partial sums is bounded.

PROBLEM 3.14. Consider the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$.

(1) Compute the first two partial sums s_1 and s_2 of this series.
(2) Explain why $s_4 = s_2 + \frac{1}{3} + \frac{1}{4} \geq \frac{3}{2} + 2(\frac{1}{4}) = \frac{4}{2}$.
(3) Explain why $s_8 = s_4 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \geq \frac{4}{2} + 4(\frac{1}{8}) = \frac{5}{2}$.
(4) Give an inductive argument to show that $s_{2^n} \geq \frac{n+2}{2}$ for each n.
(5) Explain why the series diverges.

PROBLEM 3.15. Use the comparison test (Proposition 3.17) to establish convergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^2-5n+7}$. Then use the ratio–comparison test to obtain the same result.

PROBLEM 3.16. Determine which of the following series converge. Justify your conclusions.

(1) $\sum \frac{1}{n^3-3n}$.
(2) $\sum \frac{1}{5\sqrt{n}}$.
(3) $\sum (\frac{1}{n} - \frac{1}{n+3})$.
(4) $\sum (\frac{1}{n} + \frac{1}{n+3})$.
(5) $\sum \frac{2^n}{2^n-17}$.
(6) $\sum \frac{2^n}{3^n+17}$.
(7) $\sum \frac{n}{2^n}$.

Decimal Expansions

PROBLEM 3.17. Write out the details of the proof of Proposition 3.24.

PROBLEM 3.18. Which real numbers have the following?

(1) exactly one decimal expansion
(2) exactly two decimal expansions
(3) more than two decimal expansions
(4) a terminating decimal expansion
(5) a non-terminating decimal expansion

PROBLEM 3.19. Prove Corollary 3.26.

PROBLEM 3.20. Express the repeating decimals $.\overline{23}$ and $.1\overline{35}$ as quotients of integers. (The overlined digits constitute the repeating block.) Hint: Think of geometric series.)

PROBLEM 3.21. Use long division to find the decimal expansion of $\frac{1}{37}$.

PROBLEM 3.22. Prove Proposition 3.28.

PROBLEM 3.23. Describe the decimal expansion of $\frac{1}{10^n-1}$.

PROBLEM 3.24. Express $.\overline{357} + .6\overline{89}$ as a repeating decimal.

PROBLEM 3.25. The smallest integer p satisfying Definition 3.27 is the *period* of the decimal. Characterize the fractions whose decimal expansion has period 2.

Ratio and Integral Tests

PROBLEM 3.26. Determine which of the following series converge. Justify your conclusions.

(1) $\sum \frac{n!}{(2n)!}$.
(2) $\sum \frac{n^2}{2^n}$.
(3) $\sum \frac{n!}{(n+2)!}$.
(4) $\sum \frac{n!}{(n+1)!}$.

PROBLEM 3.27. Determine which of the following series converge. Justify your conclusions.

(1) $\sum \frac{1}{\sqrt{n}}$.
(2) $\sum \frac{1}{n\sqrt{n}}$.
(3) $\sum \frac{1}{n \ln n}$.
(4) $\sum \frac{1}{n(\ln n)^2}$.

PROBLEM 3.28. Make up applications of the ratio–comparison test to each of the series of the preceding problem. Try to choose your examples so that it would be hard to apply the regular comparison test.

Series with Sign Changes

PROBLEM 3.29. Prove Proposition 3.35.

PROBLEM 3.30. Classify the following series as absolutely convergent, conditionally convergent, or divergent. Justify your conclusions.

(1) $\sum \frac{1}{n}$.
(2) $\sum \frac{1}{n^2}$.
(3) $\sum \frac{(-1)^n}{2^n}$.
(4) $\sum \frac{(-1)^n}{\ln n}$.
(5) $\sum \frac{(-1)^n n^2}{n^2+n+5}$.

PROBLEM 3.31. Use one of the convergent series from Exercise 3.27 to construct two absolutely convergent series.

PROBLEM 3.32. Use one of the divergent series from Exercise 3.27 to construct two conditionally convergent series.

PROBLEM 3.33. Prove that the sum of two absolutely convergent series is always absolutely convergent.

PROBLEM 3.34. Prove that if $\sum a_n$ converges absolutely and $\sum b_n$ converges conditionally, then the series $\sum(a_n + b_n)$ converges conditionally.

PROBLEM 3.35. Give an example of two conditionally convergent series whose sum is absolutely convergent.

Consolidation

PROBLEM 3.36. Establish the following **Root Test**; the proof is similar to that of the ratio test.

PROPOSITION 3.41. *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a series and* $\lim \sqrt[n]{|a_n|} = L$ *exists as a finite number or is infinite.*

(1) *If* $L < 1$*, then the series converges.*
(2) *If* $L > 1$*, then the series diverges.*

PROBLEM 3.37. A convergence test *fails* if it does not help you decide whether a series converges. Make up examples illustrating the failure of the following tests. No proofs are necessary, but you should describe the salient features of your examples.

(1) n'th term
(2) comparison
(3) ratio–comparison
(4) ratio
(5) integral
(6) absolute
(7) alternating

PROBLEM 3.38. Determine which of the following series converge. Identify relevant convergence tests.

(1) $\sum \frac{5}{7^{n-1}}$.
(2) $\sum(-1)^n$.
(3) $\sum \frac{1}{\ln n}$.
(4) $\sum \frac{1}{n^3}$.
(5) $\sum \frac{1}{3n-1}$.
(6) $\sum \frac{\ln n}{n}$.
(7) $\sum \frac{1}{\sqrt[n]{2}}$.
(8) $\sum \frac{\sqrt{n}}{n^2-5}$.
(9) $\sum \frac{n^2}{2^n}$.
(10) $\sum \frac{\arctan n}{n^2+1}$.
(11) $\sum \frac{\ln n}{n^2}$.
(12) $\sum \frac{1}{\sqrt{n^3+1}}$.
(13) $\sum \frac{2+\sin n}{n^2}$.
(14) $\sum \frac{2^n}{n!}$.
(15) $\sum \frac{1}{(2n-1)!}$.
(16) $\sum \frac{3^n}{n^2+1}$.
(17) $\sum \frac{n!}{9^n}$.
(18) $\sum \frac{n+1}{n^3-3n}$.
(19) $\sum \left(\frac{1}{\sqrt[3]{n}} - \frac{1}{\sqrt[3]{n+1}}\right)$.

Dirichlet's Test

PROBLEM 3.39. Explain why Dirichlet's test is a generalization of the alternating series test.

PROBLEM 3.40. Prove that if $0 < \theta < 2\pi$, then $|\sum_{k=1}^{n} \sin(k\theta)| < \operatorname{cosec}(\theta/2)$.
[Hint: Multiply by $\sin(\theta/2)$ and use a collapsing sum]

PROBLEM 3.41. Use the previous excercise to show that for all values of θ, $\sum_{n=1}^{\infty} \frac{\sin(n\theta)}{n}$ converges.

PROBLEM 3.42. Show that for θ not an even multiple of π, if b_n decreases monotonically to zero then $\sum_{n=1}^{\infty} b_n \cos(n\theta)$ converges.

CHAPTER 4

Applications to Calculus

This chapter has a dual purpose. On the one hand, it provides a quick review of several calculus concepts which you may not have been ready to appreciate the first time through. It will also provide an opportunity to apply work from Chapter 2 concerning monotone and Cauchy sequences. More specifically, we will use sequences to show that continuous functions on closed intervals attain maximum values and that their ranges do not "skip" any values.

The placement of this material is somewhat discretionary. We will not use anything from Chapter 3 here and it is possible to read most of Chapter 5 without going through the details of the present chapter. The text is brief in order to facilitate "skimming"; most interesting applications and extensions are explored in the exercises and/or left for later courses.

DEFINITION 4.1. Suppose a is a real number and S is a set of real numbers.

(1) We say that a is a *limit point* of S if for each $\epsilon > 0$ there is some number $x \in S$ satisfying $|x - a| < \epsilon$.
(2) The set of all limit points of S is called the *closure* of S.
(3) S is *closed* if each limit point of S belongs to S.

EXAMPLE 4.2. Take S to be the open interval $(0, 1)$, that is $S = \{x \in \mathbb{R} : 0 < x < 1\}$.

(1) Every member of S is a limit point of S; this is true for any set.
(2) 0 and 1 do not belong to S, but they are limit points of S.
(3) The closure of S is the interval $[0, 1]$; this is why we call $[0, 1]$ a *closed* interval.

Note the distinction between *limits* of *sequences* and *limit points* of *sets*. We will see in Proposition 4.5 however that the two concepts are closely related.

DEFINITION 4.3. Let S be a set of real numbers. A sequence (a_n) of real numbers is said to be *in* S if $a_n \in S$ for each n.

EXAMPLE 4.4. The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is a sequence in $[0, 1]$ but it is not a sequence in $(0, 1)$ because its first term fails to belong to that interval.

PROPOSITION 4.5. *Suppose a is a real number and S is a set of real numbers. Then a is a limit point of S if and only if there is a sequence in S converging to a.*

Proof. Suppose first that a is a limit point of S. For each natural number n, choose a number $a_n \in S$ with $|a_n - a| < \frac{1}{n}$; this can be done inductively. The sequence (a_n) is in S and converges to a.

For the converse, suppose that (a_n) is a sequence in S converging to a. Given $\epsilon > 0$, the definition of convergence tells us that $|a_n - a| < \epsilon$ for all sufficiently large n. In particular, $|a_n - a| < \epsilon$ for *some* $a_n \in S$, so a is a limit point of S as desired. □

Corollary 4.6. *Suppose S is a non-empty bounded set of real numbers. Then the least upper bound of S is a limit point of S.*

Proof. Write L for the least upper bound of S. Given $\epsilon > 0$, we know that $L - \epsilon$ is not an upper bound of S, so there is some number x in S with $L - \epsilon < x \leq L$. This yields $|x - L| < \epsilon$ as desired. □

Definition 4.7. Suppose D is a non–empty set of real numbers, $a \in D$, and $f : D \to \mathbb{R}$.

(1) We say f is *continuous at* a if for every $\epsilon > 0$ there is a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $x \in D$ and $|x - a| < \delta$.
(2) We say f is *continuous on* D if it is continuous at each point of D.

Proposition 4.8. *Suppose $a \in D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. Then the following are equivalent.*

(1) *f is continuous at a.*
(2) *If (a_n) is any sequence in D converging to a, then the sequence $(f(a_n))$ converges to $f(a)$.*

Proof. 1) $\Longrightarrow$ 2). Suppose that f is continuous at a, that (a_n) is a sequence in D converging to a, and that $\epsilon > 0$ is given. Use the definition of continuity to find $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$. Next, find N so that $|a_n - a| < \delta$ whenever $n \geq N$. Then $n \geq N \Longrightarrow |f(a_n) - f(a)| < \epsilon$ as desired.

2) $\Longrightarrow$ 1). We argue contrapositively, assuming that 1) fails. Thus for *some* $\epsilon > 0$, there are points in D arbitrarily close to a whose distances from $f(a)$ equal or exceed ϵ. In particular, for each $n \in \mathbb{N}$, we can find a number $a_n \in D$ with $|a_n - a| < \frac{1}{n}$, but $|f(a_n) - f(a)| \geq \epsilon$. Thus (a_n) is a sequence in D converging to a, but the image sequence $(f(a_n))$ cannot converge to $f(a)$, and we have shown that 2) fails. □

Proposition 4.9. *Suppose $f, g : D \to \mathbb{R}$ are continuous at some number $c \in D$.*

(1) *$f + g$ is continuous at c.*
(2) *$f \cdot g$ is continuous at c.*
(3) *If $g(c) \neq 0$, then $\frac{f}{g}$ is continuous at c.*
(4) *All polynomial functions are continuous on $\mathbb{R}$.*
(5) *All rational functions are continuous throughout their domains.*

HINT. One approach is to mimic the *proof* of Proposition 2.19. It is easier and neater to use Proposition 4.8 to set up an application of the *result* of Proposition 2.19. □

THEOREM 4.10. *(Intermediate Value Theorem) Suppose* $f : [a, b] \to \mathbb{R}$ *is continuous on its domain,* $f(a) < 0$*, and* $f(b) > 0$*. Then* $f(c) = 0$ *for some* $c \in (a, b)$*.*

PROOF. Set $S = \{x \in [a, b] : f(x) \leq 0\}$. Note that S is non–empty and bounded so it has a least upper bound c. By definition of least upper bound, for each n, we can find a number $a_n \in S$ satisfying $c - \frac{1}{n} < a_n \leq c$. Proposition 4.8 tells us that $\lim f(a_n) = f(c)$. On the one hand, the fact that $f(a_n) \leq 0$ for each n leads to the conclusion that $f(c) \leq 0$. On the other hand, since c is an upper bound of S, we must have $f(c + \frac{1}{n}) > 0$ for sufficiently large n and this forces $f(c) \geq 0$. Thus $f(c) = 0$ as desired. □

EXAMPLE 4.11. The Intermediate Value Theorem can be used to show that certain equations have solutions. For example, the polynomial $f(x) = x^2 - 2$ is continuous everywhere and satisfies $f(1) < 0 < f(2)$. Thus the equation $f(x) = 0$ has a root $x \in (1, 2)$; in other words, $\sqrt{2}$ exists as a real number.

EXAMPLE 4.12. The Intermediate Value Theorem can also be used to solve inequalities. Consider, for example the inequality

$$\frac{x^3 - 2x^2}{x - 4} \leq 0.$$

To solve (4.1), introduce the rational function $f(x) = \frac{x^3 - 2x^2}{x - 4}$. The Intermediate Value Theorem tells us that the only places where f *could* change sign are points where it is zero ($x = 0, 2$) and points where it fails to be continuous ($x = 4$). Checking signs of $f(-1), f(1), f(3)$, and $f(5)$, we thus see that

$$\begin{aligned} f(x) > 0 \qquad & \text{for } x \in (-\infty, 0) \cup (0, 2) \cup (4, \infty), \text{while} \\ f(x) < 0 \qquad & \text{for } x \in (2, 4). \end{aligned}$$

Since $f(2) = 0$, while $4 \notin \text{Domain}(f)$, we conclude that the solution to (4.1) consists of the half–open interval $[2, 4)$.

It is usual to organize this work by plotting a *sign chart* for f, i.e. representing zeros and discontinuities of f by dots and hollow circles respectively on a number line, and then recording the sign of f on each intervening interval.

Recall that a function $f : D \to \mathbb{R}$ is said to *attain a maximum value at* $c \in D$ if $f(x) \leq f(c)$ for all $x \in D$.

THEOREM 4.13. *Suppose* $f : [a, b] \to \mathbb{R}$ *is continuous on its domain. Then* f *attains a maximum value at some* $c \in [a, b]$*.*

Proof. We first show that the range of f is bounded above. If not, then for each n, we could find a number $a_n \in [a, b]$ with $f(a_n) > n$. Applying Theorem 2.33, we could then find a convergent subsequence of (a_n); replacing the original sequence by this subsequence if necessary, we may assume that $\lim a_n = c$ for some $c \in [a, b]$. But Proposition 4.8 would then yield $\lim f(a_n) = f(c)$ whence the sequence $(f(a_n)$ would be bounded in view of Proposition 2.15. This conclusion is incompatible with the choice of the (a_n) and completes our proof by contradiction that the range of f is indeed bounded.

Write L for the least upper bound of the range of f. By definition of least upper bound, for each natural number n, we can find a number $a_n \in [a, b]$ with $L - \frac{1}{n} < f(a_n) \leq L$. Applying Theorem 2.33 and dropping down to an appropriate subsequence if necessary, we may as well assume that $\lim a_n = c$ for some $c \in [a, b]$. But then Proposition 4.8 yields $L = \lim f(a_n) = f(c)$ and we have shown that $L = f(c)$ is the maximum value of f as desired. □

In keeping with the brief nature of this chapter, we leave it to the reader to review the derivative concept from his/her favorite calculus book (or the exercises at the end of this chapter). In particular, the reader is assumed to feel comfortable with the fact that the number c guaranteed by Theorem 4.13 must either be an endpoint of $[a, b]$, a place where $f'(c) = 0$, or a place where $f'(c)$ fails to exist.

Theorem 4.14. *(Rolle's Theorem) Suppose $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and satisfies $f(a) = f(b)$. Then $f'(c) = 0$ for some $c \in (a, b)$.*

Proof. It follows easily from Theorem 4.13 that f attains maximum *and* minimum values. If at least one of these occurs at some $c \in (a, b)$, then $f'(c) = 0$ by the preceding paragraph. On the other hand, if the extreme values of f are both attained at endpoints, then f is constant and so $f'(c) = 0$ for all $c \in (a, b)$ □

Example 4.15. We have seen how the Intermediate Value Theorem can be used to show that an equation has *at least* one root. Rolle's Theorem shows some equations have *at most* one root. When both theorems can be applied, it is possible to conclude that an equation has a *unique* root.

Consider, for example the equation $f(x) = x^3 + x + 1 = 0$. Since f is differentiable everywhere, and $f'(x)$ is never zero, Rolle's Theorem rules out having $f(a) = f(b)$ for distinct a and b. Since $f(-1)$ and $f(0)$ have opposite signs, the Intermediate Value Theorem also applies to show that the equation has a unique root.

Corollary 4.16. *(Generalized Mean Value Theorem) Suppose f and g are real–valued functions which are continuous on some closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is some number $c \in (a, b)$ satisfying*

$$f'(c)[g(b) - g(a)] = g'(c)[f(b) - f(a)].$$

PROOF. Set $h(x) = [f(x)-f(a)][g(b)-g(a)]-[g(x)-g(a)][f(b)-f(a)]$. Note that h satisfies the hypotheses of Rolle's Theorem. The conclusion of that theorem thus yields a number $c \in (a,b)$ with $h'(c) = 0$. Substitution and transposition shows this is the desired result. □

COROLLARY 4.17. *(Mean Value Theorem) Suppose f is a real–valued function which is continuous on some closed interval $[a,b]$ and differentiable on the open interval (a,b). Then there is some number $c \in (a,b)$ satisfying*

$$f'(c) = \frac{f(b)-f(a)}{b-a}.$$

PROOF. Apply the preceding corollary with $g(x) = x$. □

The Mean Value Theorem has a nice geometric interpretation concerning secant and tangent lines associated with the graph of f. It also has a physical interpretation: if you travel 60 miles in one hour, your *instantaneous* velocity will be 60 mph at least once during the trip. The next, admittedly technical, result will be used in the next chapter.

COROLLARY 4.18. *Suppose h has derivatives of all orders on some open interval D containing a and b and $h^{(i)}(a) = 0$ for $0 \le i \le n$. Then there is some number c between a and b satisfying*

$$h(b) = \frac{h^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

PROOF. We argue by induction on n. When $n = 0$ this is just the Mean Value Theorem with $f = h$.

For the inductive step, assume the result holds for $n = k$ and suppose that h satisfies $h^{(i)}(a) = 0$ for $0 \le i \le k+1$. Applying the generalized mean value theorem to $f = h$ and $g(x) = (x-a)^{n+1}$ yields a number $d \in (a,b)$ with

$$h(b) = \frac{h'(d)}{(n+1)(d-a)^n}(b-a)^{n+1}.$$

On the other hand, applying the inductive hypothesis to h' with d in place of b yields a number $c \in (a,d)$ with

$$h'(d) = \frac{h^{(n+1)}(c)}{(n)!}(d-a)^n.$$

Substituting (4.3) in (4.2) gives the desired result. □

Exercises

PROBLEM 4.1. Find the closures of the following sets of real numbers. No proofs are required.

(1) $\mathbb{R}$,
(2) $\emptyset$ (the empty set),
(3) $\{x \in \mathbb{R} : x \ge 0\}$,
(4) $\{x \in \mathbb{R} : x > 0\}$,

(5) $\mathbb{N}$ (the natural numbers),
(6) $\mathbb{Q}$,
(7) $\{\frac{1}{n} : n \in N\}$,
(8) $\{\frac{n+1}{n} : n \in N\}$.

PROBLEM 4.2. In this problem, we examine operations on closed sets.

(1) Prove that if A and B are closed, then the union $A \cup B$ is also closed.
(2) Prove that the union of any finite collection of closed sets is closed.
(3) Give an example of a(n infinite) collection of closed sets whose union is not closed.
(4) Prove that the intersection of any collection of closed sets is closed.

PROBLEM 4.3. A set S of real numbers is said to be *open* if for each $a \in S$, there is some $\epsilon > 0$ such that $x \in S$ whenever $|x - a| < \epsilon$. Prove that a set $S \subset \mathbb{R}$ is open if and only if the complementary set $\sim S$ is closed.

PROBLEM 4.4. Give examples of the following:

(1) a closed set which is not open,
(2) an open set which is not closed,
(3) a set which is both open and closed,
(4) a set which is neither open nor closed,

PROBLEM 4.5. In this problem, we examine operations on open sets.

(1) Prove that if A and B are open, then the union $A \cap B$ is also open.
(2) Prove that the intersection of any finite collection of open sets is open.
(3) Give an example of a(n infinite) collection of open sets whose intersection is not open.
(4) Prove that the union of any collection of open sets is open.

PROBLEM 4.6. Prove the first part of Proposition 4.9 by mimicking the *proof* of Proposition 2.19.

PROBLEM 4.7. Complete the proof of Proposition 4.9 by using Proposition 4.8 and the *results* of Proposition 2.19.

PROBLEM 4.8. Suppose $D \subset \mathbb{R}$, $f : D \to \mathbb{R}$, $L \in \mathbb{R}$, and a is a limit point of D. Give an ϵ–δ type definition of the limit concept $\lim_{x \to a} f(x) = L$. (Caution: The value of $f(a)$ should be irrelevant.)

PROBLEM 4.9. State an analogue of Proposition 4.8 to go with the limit concept of the preceding problem.

PROBLEM 4.10. Recall the definition of what it means for a function f to be differentiable at a number $a \in \mathbb{R}$ and prove that differentiability at a implies continuity at a.

PROBLEM 4.11. Reread the proofs of various differentiation rules in your favorite calculus book. Then write a one paragraph essay on your appreciation of such matters.

PROBLEM 4.12. Find a discontinuous function $f : [0, 1] \to \mathbb{R}$ which does not satisfy the conclusion of the Intermediate Value Theorem. (The domain of f should not omit any points in $[0, 1]$).

PROBLEM 4.13. Find a discontinuous function $f : [0, 1] \to \mathbb{R}$ which does satisfy the conclusion of the Intermediate Value Theorem.

PROBLEM 4.14. Use the Intermediate Value Theorem to prove that every positive real number has a real square root.

PROBLEM 4.15. Solve the inequality $\frac{x^4-10x^2+9}{x^2-x+2} \geq 0$.

PROBLEM 4.16. Prove that every continuous function on a closed interval $[a, b]$ attains a minimum value.

PROBLEM 4.17. Give an example of a continuous function $f : \mathbb{R} \to \mathbb{R}$ which does not attain a maximum value.

PROBLEM 4.18. Give an example of a continuous function $f : (0, 1) \to \mathbb{R}$ which attains a maximum value, but not a minimum value.

PROBLEM 4.19. Give an example of a continuous function $f : (0, 1] \to \mathbb{R}$ which attains neither a maximum nor a minimum value.

PROBLEM 4.20. Suppose $f : \mathbb{R} \to \mathbb{R}$. Give a precise definition of the concept $\lim_{x\to\infty} f(x) = \infty$.

PROBLEM 4.21. Which polynomials p satisfy $\lim_{x\to\infty} p(x) = \infty$? Which satisfy $\lim_{x\to-\infty} p(x) = -\infty$?

PROBLEM 4.22. Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \infty$. Prove that f attains a minimum value.

PROBLEM 4.23. Characterize the class of polynomials whose ranges exhaust $\mathbb{R}$.

PROBLEM 4.24. Illustrate the meaning of Rolle's Theorem graphically.

PROBLEM 4.25. Prove that the equation $x^3 + 3x = \sin x$ has a unique solution.

PROBLEM 4.26. Prove that the equation $x^3 + x = \sin x$ has a unique solution.

PROBLEM 4.27. Give examples to show how Rolle's Theorem can fail if any of its hypotheses are violated:

(1) differentiability throughout the open interval,
(2) continuity at the endpoints,
(3) equality of $f(a)$ and $f(b)$.

PROBLEM 4.28. Give an example to show that the number c guaranteed by Rolle's Theorem need not be unique.

PROBLEM 4.29. Illustrate the meaning of the Mean Value Theorem graphically.

PROBLEM 4.30. Explain the geometric meaning of the auxiliary function h used in the proof of Theorem 4.16 in the special case $g(x) = x$.

PROBLEM 4.31. Use Corollary 4.16 to prove the following version of l'Hôpital's rule.

PROPOSITION 4.19. *Suppose f and g are real–valued functions which are differentiable in some open interval containing the real number a and satisfy $f(a) = f(b) = 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

as long as the latter limit exists as a finite number.

PROBLEM 4.32. State and prove Corollary 4.18 in the special cases $n = 1$ and $n = 2$ without appealing to induction.

CHAPTER 5

Taylor's Theorem

1. Statement and Proof

The underlying theme in differential calculus is using tangent lines to gain local information about relatively complicated functions. Taylor's Theorem refines this idea by using polynomials of arbitrary degree in place of the first degree polynomials describing tangent lines.

EXAMPLE 5.1. For a concrete illustration, consider the sine function $f(x) = \sin x$. The crudest approximation is the constant function $p_0(x) = \sin 0 = 0$. There are three senses in which p_0 approximates f near 0.

(1) (at zero) $f(0) = p_0(0)$ by construction.
(2) (approaching zero) $\lim_{x\to 0} f(x) - p_0(x) = 0$ by continuity.
(3) (error quantification) For each $x \in \mathbb{R}$, the Mean Value Theorem tells us there is some number c between 0 and x such that the error $f(x) - p_0(x) = f'(c)(x - 0) = (\cos c)x$.

To see how (3) might be useful, suppose we need information about $\sin .01$. Since .01 is close to 0 we would expect $\sin .01$ to be near zero, and (3) tells us that

$$|\sin(.01)| = |f(.01) - p_0(.01)| = |(\cos c)(.01)| \leq .01. \tag{5.1}$$

The key point here is that even though we don't know what c is, our general knowledge of the cosine function tells us that $|\cos c| \leq 1$.

It should also be emphasized that the information provided by (3) deteriorates quickly as we move away from the origin, e.g., the fact that $|\sin 3| \leq 3$ is not particularly enlightening. The problem, of course, is that the sine function is not constant: geometrically we should do better with a tangent line than with a horizontal line, so we use the first degree polynomial $p_1(x) = x$. The above conditions improve to

(1) (at zero) $f(0) = p_1(0)$ and $f'(0) = p_1'(0)$ by construction.
(2) (approaching zero) L'hopital's rule tells us that
$$\lim_{x\to 0} \frac{f(x) - p_1(x)}{x} = \lim_{x\to 0} \frac{\sin x - x}{x} = \lim \frac{\cos x - 1}{1} = 0,$$
i.e., not only does the error $f(x) - p_1(x)$ go to zero as $x \to 0$, but it does so *faster than* x.
(3) (error quantification) Two applications of the Generalized Mean Value Theorem 4.16 tell us that for each $x \in \mathbb{R}$, there is a number

c between 0 such that the error

$$f(x) - p_1(x) = \frac{f''(c)}{2}x^2 = \frac{-\sin c}{2}x^2.$$

In particular, Display 5.1 improves to

$$|\sin(.01) - .01| = |f(.01) - p_1(.01)| = |\frac{\sin c)}{2}(.01)^2| \leq .00005,$$

that is, $\sin(.01)$ is between .00995 and .01005.

Taylor's Theorem generalizes this analysis is two respects: higher derivatives are taken into account, and approximations can be centered at any real number a.

DEFINITION 5.2. Let a be a real number, n a non-negative integer, and suppose f is a function having derivatives of all orders at a.

(1) The polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is called the *n'th order Taylor polynomial of f about a.*

(2) The difference $r_n(x) := f(x) - p_n(x)$ is called the *n'th remainder* of *f about a.*

(3) The infinite series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{5.2}$$

is called the *Taylor series of f about a.*

(4) Taylor polynomials, remainders, and series about 0 are also referred to as *Maclaurin* polynomials, remainders, and series, resprctively.

There are two ways to look at Display 5.2. We can think of it as a series of *functions*. From this perspective, the n'th partial sum of the series coincides with the n'th order Taylor polynomial of f about a, so that the Taylor series of f about a simultaneously encodes all of its Taylor polynomials about a.

Alternatively, we can think of a Taylor series as a "series-valued function", that is, for each fixed $x \in \mathbb{R}$, it gives us a *numerical* series which may or or may not converge. We are interested in determining when such a series converges to $f(x)$ itself.

EXAMPLE 5.3. Take $a = 0$ and $f(x) = x^2 + 3x + 5$. We note that $f(0) = 5, f'(0) = 3, f''(0) = 2$, and all higher order derivatives of f are zero. Thus we have the Maclaurin polynomials $p_0(x) = 5, p_1(x) = 3x + 5$, and $p_n(x) = x^2 + 3x + 5$ for all $n \geq 2$. In particular, f is its own Maclaurin series.

As suggested by Example 5.1, Maclaurin polynomials should be regarded as approximations of f near 0. p_0 is the crudest of these: it is designed to agree with f at zero, but because it is constant, its accuracy deteriorates quickly as x leaves zero. $y = p_1(x)$ is the equation of the line tangent to f at 0. Thus it is a better approximation of f near zero than p_0. p_2 is even better (perfect in this case) because by taking account of $f''(0)$, it corrects for the fact that f is not linear.

EXAMPLE 5.4. Take $a = 2$ and $f(x) = x^2 + 3x + 5$. We note that $f(2) = 15, f'(2) = 7, f''(2) = 2$, and all higher order derivatives of f are zero. Thus we have the Taylor polynomials $p_0(x) = 15, p_1(x) = 7(x-2)+15$, and $p_k(x) = (x-2)^2 + 7(x-2) + 15$ for all $k \geq 2$. Direct computation shows that f coincides with p_2.

The zero'th and first degree Taylor polynomials in this example are not the same as in the previous example; this was to be expected since we are now approximating f near 2 rather near 0. The fact that p_2 agrees with f in both examples should be regarded as reflecting the simple nature of f.

EXAMPLE 5.5. Take $a = 0$ and $f(x) = e^x$. Since $f^{(n)}(0) = 1$ for all n, the Maclaurin series for f is given by $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Here is the desired generalization of Example 5.1. To avoid writing too many primes, we denote the k'th order derivative of f by $f^{(k)}$; in particular, $f^{(0)}$ is f itself. Also, recall that $0! = 1$.

THEOREM 5.6. *(Taylor's Theorem) Let a be a real number, and suppose f is a function having derivatives of all orders in some open interval J containing a. Then for each non-negative integer n, the Taylor polynomial p_n is a good approximation of f in the following senses:*

1. *(at a) For each $k \leq n$, we have $r_n^{(k)}(a) = 0$, that is, f and each of its first n derivatives agree with those of p_n at a.*
2. *(approaching a) $\lim_{x\to a} \frac{r_n(x)}{(x-a)^n} = 0$, that is, as x approaches a, the n'th remainder approaches zero faster than $(x-a)^n$.*
3. *(error quantification) Given x in J, there is some number c between a and x satisfying*

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{5.3}$$

PROOF. Fix n and set $h(x) = r_n(x) = f(x) - p_n(x)$.

An induction argument shows that $h^{(k)}(a) = 0$ for $0 \leq k \leq n$, thereby establishing (1).

Given $x \in J$, apply Corollary 4.18 to find a number c between a and x with

$$r_n(x) = h(x) = \frac{h^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}. \tag{5.4}$$

Dividing both sides of this equation by $(x-a)^n$ and taking the limit as x approaches a yields (2).

Finally, the $(n+1)$'st derivative of p_n is identically zero, so $h^{(n+1)}(c) = f^{(n+1)}(c)$ and (5.4) reduces to Equation 5.3, thereby establishing (3). □

Part(2) of the Theorem will be used in Section 3 to provide shortcut methods for finding Maclaurin series of functions like $\sin x^2$ whose successive derivatives quickly get messy. The rest of the present section will focus on applications of the third part of the Theorem.

EXAMPLE 5.7. Use the third order Maclaurin polynomial for $f(x) = \cos x$ to approximate $\cos .1$ and discuss the accuracy of this approximation.

SOLUTION. We have $f(0) = 1, f'(0) = 0, f''(0) = -1, f^{(3)}(0) = 0$, and $f^{(4)}(x) = \cos x$. Thus we get

$$p_3(x) = 1 - \frac{x^2}{2!} \text{ with remainder } r_3(x) = \frac{\cos c}{4!}x^4.$$

Thus we approximate $\cos .1$ by $p_3(.1) = .1 - \frac{.01}{2} = .095$ and can assert that the error of our approximation is at most $|r_3(.1)| \leq \frac{.0001}{24} < .000005$. □

EXAMPLE 5.8. What Maclaurin polynomial could be used to approximate $\sqrt{\mathrm{e}}$ to to within .001 ?

SOLUTION. Take $f(x) = \mathrm{e}^x$. We need to find n with $|r_n(\frac{1}{2})| < .001$. Applying Taylor's Theorem, we get

$$\left|r_n(\frac{1}{2})\right| = \frac{\mathrm{e}^c}{(n+1)!2^{n+1}} < \frac{1}{(n+1)!2^n},$$

because we know that $\mathrm{e} < 4$ and $c < \frac{1}{2}$. Thus we need to choose n so that $\frac{1}{(n+1)!2^n} < \frac{1}{1000}$, i.e., $(n+1)!2^n > 1000$. This is done by trial and error. $n = 3$ is too small since $(4!)2^3 = 384$ but $n = 4$ will definitely work. Thus the right polynomial to use is $p_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{16}$ □

EXAMPLE 5.9. Use a suitable second order Taylor polynomial to approximate $\sqrt{99}$, and discuss the error in your approximation.

SOLUTION. . We take $f(x) = \sqrt{x}$ and $a = 100$; the choice of a is based on two facts: we know its square root and it's close to 99. (We could just as well have used a Maclaurin polynomial of $g(x) = \sqrt{x+100}$.) We have $p_2(x) = 10 + \frac{1}{20}(x-100) - \frac{1}{8000}(x-100)^2$, so we estimate $\sqrt{99} \sim p_2(99) = 10 - \frac{401}{8000} = 9.94875$. The error is given by $|r_2(99)| = \frac{1}{16c^{\frac{5}{2}}}$ where c is somewhere between 99 and 100. The worst scenario is when c is close to 99. Of course, we don't know how to take the $\frac{5}{2}$ power of 99, but we do know that it exceeds $64^{\frac{5}{2}} > 3(10^5)$. Thus $|\sqrt{99} - 9.94875| < 10^{-5}$. In fact, MAPLE gives $\sqrt{99} \sim 9.9498743$, well within our tolerance. □

2. Taylor Series

The alert reader will have noticed that we have avoided saying that functions are "equal" to their Taylor series. Here is a silly example to show why.

EXAMPLE 5.10. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} 0, & |x| < 2 \\ 17, & |x| \geq 2 \end{cases}$. Then the Maclaurin series of f is identically zero, but that doesn't agree with f when $|x| \geq 2$.

In general, we write $f(x) \sim \sum \frac{f^{(n)}(a)}{n!}(x-a)^n$ to denote the fact that the expression on the right is the Taylor series of f.

On the other hand, we write

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n, \quad x \in D \tag{5.5}$$

to signify the fact that the *numerical* series on the right converges to $f(x)$ for each $x \in D$. When D is an open interval, we also express Equation 5.5 by saying f is *analytic on D*.

Thus in the last example, it is correct to write $f \sim 0$, or $f(x) = 0$, for $|x| < 2$, but it would be wrong to write $f = 0$ unrestrictedly.

We turn to less pathological situations. By definition, the n'th remainder of f about a is the difference between f and the n'th partial sum of its Taylor series about a. Thus we have the following.

PROPOSITION 5.11. *In order that the Taylor series for a function f converge to $f(x)$ at a real number x it is necessary and sufficient that* $\lim_{n\to\infty} r_n(x) = 0$.

EXAMPLE 5.12. Find the Maclaurin series for $f(x) = \cos x$ and show that it converges to $f(x)$ for all real numbers x.

SOLUTION. Since all odd-order derivatives of the cosine function vanish at 0, we only need to compute even-order derivatives. We have $f^{(0)}(0) = \cos 0 = 1, f^{(2)}(0) = -\cos 0 = -1$ and in general, $f^{(2n)}(0) = (-1)^n$, so that the Maclaurin series for f is given by

$$\sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!}x^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots.$$

For any $x \in \mathbb{R}$, Part (3) of Taylor's Theorem yields

$$|r_n(x)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1} \right| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

We know the series $\sum \frac{|x|^n}{n!}$ converges by the ratio test, so $\lim_{n\to\infty} r_n(x) = 0$ by the n'th term test. In other words, $\lim_{n\to\infty} p_n(x) = f(x)$, as required. □

Only now, *after* showing the Maclaurin series for cos does indeed converge to the original function, is it legitimate to write

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}, \tag{5.6}$$

and to say that f is analytic throughout $\mathbb{R}$. Similar (but simpler) reasoning shows the exponential function of Example 5.5 is analytic everywhere.

EXAMPLE 5.13. Consider the function $f(x) = \frac{1}{1-x}$. Direct computation gives $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$ so the Maclaurin series of f is the geometric series $\sum_{n=0}^{\infty} x^n$. By our analysis of geometric series in Proposition 3.12 we know this function is analytic on all of $(-1, 1)$, i.e.,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \qquad -1 < x < 1. \tag{5.7}$$

This is an example where the error estimate in Taylor's Theorem does not give us complete information. Indeed, that error estimate is $r_n(x) = \dfrac{x^{n+1}}{(1-c)^{n+2}}$ for some c between 0 and x. It's easy to show that these remainders approach zero when $-1 < x < \frac{1}{2}$, but when $x > \frac{1}{2}$, we have no way of knowing $c \neq \frac{1}{2}$, in which case $\frac{x^{n+1}}{(1-c)^{n+2}}$ would not approach 0.

3. Operations on Taylor Polynomials and Series

Throughout this section f has derivatives of all orders in some open interval $(-b, b)$ containing 0. By definition, the n'th order Maclaurin polynomial p_n of f is given by

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k \tag{5.8}$$

and this is related to the n'th Maclaurin remainder of f by

$$f(x) = p_n(x) + r_n(x), \quad |x| < b \tag{5.9}$$

In this section, we perform various operations on these equations to give us new Taylor polynomials. As a mnemonic device, each application is given a brief title.

PROPOSITION 5.14. *(Differentiation) The n'th order Maclaurin polynomial of f' is the derivative of the $(n+1)'$st order Maclaurin polynomial of f. The same is true for the corresponding remainders on the interval $|x| < b$.*

PROOF. Write $g = f'$. Then successive derivatives of g are always one behind of those of f. In particular, $g^{(k)}(0) = f^{(k+1)}(0)$ for each non-negative integer k, so the n'th order Maclaurin polynomial of g is given by

$$\sum_{k=0}^{n} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{f^{(k+1)}(0)}{k!} x^k,$$

which we recognize from Equation 5.8 as the derivative of $p_{n+1}(x)$.

On the other hand, differentiating Display 5.9, we obtain

$$g(x) = f'(x) = p'_{n+1}(x) + r'_{n+1}(x), \quad , |x| < b$$

Now that we have identified p'_{n+1} to be the n'th order Maclaurin polynomial of g, this equation forces r'_{n+1} to be the n'th Maclaurin remainder of g. □

COROLLARY 5.15. *The Maclaurin series for f' can be obtained from the Maclaurin series of f by termwise differentiation. If f is analytic on $|x| < b$, then the derivative f' is analytic on that interval as well.*

PROOF. This follows from Proposition 5.14 because Maclaurin polynomials are partial sums of the corresponding Maclaurin series. □

The remaining Propositions in this section have similar corollaries.

EXAMPLE 5.16. Since the fourth order Maclaurin polynomial of $\frac{1}{1-x}$ is $1+x+x^2+x^3+x^4$, the Proposition identifies $1+2x+3x^2+4x^3$ as the third order Maclaurin polynomial of $\frac{1}{(1-x)^2}$. This can also be checked directly.

Moreover, differentiating Equation 5.7 term by term, we conclude

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=1}^{\infty} nx^{n-1}, \qquad -1 < x < 1.$$

PROPOSITION 5.17. *(Integration) Define a function F by setting $F(x) = \int_0^x f(t)dt$, for $|x| < b$ and suppose n is a natural number.*

1. *The n'th order Maclaurin polynomial of F is given by $P_n(x) = \int_0^x p_{n-1}(t)dt$.*
2. *For $|x| < b$, the n'th Maclaurin remainder of F is given by $R_n(x) = \int_0^x r_{n-1}(t)dt$.*
3. *For each $|x| < b$, there is some c between 0 and x so that $R_n(x) = xr_{n-1}(c)$.*

PROOF. By the Fundamental Theorem of Calculus, F is the unique antiderivative of f which vanishes at 0. Thus successive derivatives of F are always one ahead of those of f. In particular, $F^{(k)}(0) = f^{(k-1)}(0)$ for each positive integer k, so the n'th order Maclaurin polynomial of F is given by

$$\sum_{k=0}^{n} \frac{F^{(k)}(0)}{k!}x^k = \sum_{k=1}^{n} \frac{f^{(k-1)}(0)}{k!}x^k,$$

which we recognize from Equation 5.8 as $\int_0^x p_{n-1}(t)dt$.

On the other hand, integrating Display 5.9, we obtain

$$F(x) = \int_0^x f(t)dt + \int_0^x p_{n-1}(t)dt + \int_0^x r_{n-1}(t)dt, \quad ,|x| < b$$

Now that we have identified $\int_0^x f(t)dt$ as $P_n(x)$, the rightmost term in the last display must be $R_n(x)$. □

EXAMPLE 5.18. Termwise integration of Equation 5.6 yields the Maclaurin series of sin and shows that series converges to $\sin x$ throughout the reals:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}. \tag{5.10}$$

EXAMPLE 5.19. Since the third order Maclaurin polynomial of $\frac{1}{1-x}$ is $1 + x + x^2 + x^3$, the Proposition identifies $P_4(x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ as the fourth order Maclaurin polynomial of $F(x) = -\ln(1-x)$. This can be checked by computing the fourth order Maclaurin polynomial directly from Definition 5.2. In any case, $P_4(\frac{1}{2}) = \frac{131}{192} \sim .6823$ should approximate $-\ln(\frac{1}{2}) = \ln 2$.

To assess the accuracy of this approximation, recall from the proof of Proposition 3.12, that for $x \neq 1$, we have $1+x+x^2+x^3 = \frac{1-x^4}{1-x}$. This means the third Maclaurin remainder of $\frac{1}{1-x}$ is $r_3(x) = \frac{x^4}{1-x}$. Thus $r_3(t) \leq 2t^4$ for $0 \leq t \leq \frac{1}{2}$, the fourth Maclaurin remainder of F satisfies $R_4(x) = \int_0^{\frac{1}{2}} r_3(t)dt \leq \int_0^{\frac{1}{2}} 2t^4 dt = \frac{1}{80} \sim .0125$. The true value of $\ln 2 \sim .6931$ so our error bound is fairly tight. Direct use of Taylor's remainder formula Theorem 5.6.3 and Proposition 5.17.4 would have given much poorer error bounds.

The next application provides a tool for passing between Maclaurin polynomials and Taylor polynomials based at other points.

PROPOSITION 5.20. *(Translation) Let $a \in \mathbb{R}$, and define a new function g by $g(x) = f(x-a)$. Then the n'th order Taylor polynomial of g about a can be obtained by substituting $x - a$ for x in the n'th order Maclaurin polynomial of f. The same is true for n'th order remainders on the interval $|x| < b$.*

PROOF. We have $g(x) = f(x-a)$ by hypothesis. Since $D_x(x-a) = 1$, the chain rule yields $g'(x) = f'(x-a)$. In fact, differentiating repeatedly yields $g^{(k)}(x) = f^{(k)}(x-a)$ for each non-negative integer k. In particular, taking $x = a$, we get $g^{(k)}(a) = f^{(k)}(0)$. Substituting in Equation 5.8, we get

$$p_n(x-a) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}(x-a)^k = \sum_{k=0}^{n} \frac{g^{(k)}(a)}{k!}(x-a)^k,$$

which we recognize as the n'th order Taylor polynomial of g about a.

On the other hand, substituting $x - a$ for x in Display 5.9, we obtain

$$g(x) = f(x-a) = p_n(x-a) + r_n(x-a), \quad |x-a| < b$$

Now that we have identified $p_n(x-a)$ to be the n'th order Taylor polynomial of g about a, this equation forces $r_n(x-a)$ to be the n'th remainder of g about a. □

EXAMPLE 5.21. It follows from Proposition 5.20 that the third order Maclaurin polynomial of $f(x) = \sqrt{x + 100}$ is the same as the third order Taylor polynomial of $g(x) = f(x - 100) = \sqrt{x}$ about $a = 100$. This is easily checked directly.

Consider the problem of finding the fourth order Maclaurin polynomial of the function $f(x) = x^2 \cos x$. The product rule makes repeated differentiation rather messy. On the other hand, we already know that $1 - \frac{x^2}{2}$

is the second degree Maclaurin polynomial of $\cos x$ and is tempting to just multiply by x^2 to get the desired Maclaurin polynomial for f. The way we see that's right is by applying Part (2) of Taylor's Theorem.

PROPOSITION 5.22. *(Maclaurin Characterization) Suppose a function h has derivatives of all orders in some open interval about 0 and n is a non-negative integer. If a polynomial q has degree at most n and satisfies $\lim_{x\to 0} \frac{h(x)-q(x)}{x^n} = 0$, then q* **must be** *the n'th order Maclaurin polynomial of f about a.*

PROOF. Write q_n for the n'th order Maclaurin polynomial of h. Part (2) of Taylor's Theorem tells us that $\lim_{x\to 0} \frac{h(x)-q_n(x)}{x^n} = 0$. Subtraction then yields $\lim_{x\to 0} \frac{q(x)-q_n(x)}{x^n} = 0$. Since the degree of $q - p_n$ is at most n, this forces $q - q_n \equiv 0$. □

COROLLARY 5.23. *(Monomial Multiplication) Suppose k is a positive integer and define a function F by $F(x) = x^k f(x)$, for $|x| < b$. For each natural number n,*

(1) *The (n+k)'th order Maclaurin polynomial of F is given by the formula $P_{n+k}(x) = x^k p_n(x)$.*
(2) *For $|x| < b$, the (n+k)'th order Maclaurin remainder of F is given by $R_{n+k}(x) := x^k r_n(x)$.*

PROOF. Canceling a factor of x^k from numerator and denominator we obtain

$$\lim_{x\to 0} \frac{F(x) - x^k p_n(x)}{x^{n+k}} = \lim_{x\to 0} \frac{f(x) - p_n(x)}{x^n} = 0$$

by Part (2) of Taylor's Theorem. Thus Proposition 5.22 tells us that $x^k p_n(x)$ *must be* the $(n+k)'th$ order Maclaurin polynomial of F. That establishes (1). (2) follows by multiplying both sides of Display 5.9 by x^k. □

EXAMPLE 5.24. From Example 5.18, we learn that $x^2 \sin x$ has Maclaurin series $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+3}$ and this series converges to $x^2 \sin x$ throughout the reals.

COROLLARY 5.25. *(Monomial Substitution) Suppose k is a positive integer and d is a real number. Define a function F by $F(x) = f(dx^k)$, for $|x| < b$. For each natural number n,*

(1) *The (nk)'th order Maclaurin polynomial of F is given by $P_{nk}(x) = p_n(dx^k)$.*
(2) *For $|x| < b$, the (nk)'th order Maclaurin remainder of F is given by $R_{nk}(x) := r_n(dx^k)$.*

PROOF. Substituting $y = dx^k$ we obtain

$$\lim_{x\to 0} \frac{F(x) - p_n(dx^k)}{x^{nk}} = \lim_{y\to 0} d^n \frac{f(y) - p_n(y)}{y^n} = 0$$

by Part (2) of Taylor's Theorem. Thus Proposition 5.22 shows that $p_n(dx^k)$ *must be* the $(nk)'th$ order Maclaurin polynomial of F. That establishes (1). (2) follows by substituting dx^k for x in both sides of Display 5.9. $\square$

EXAMPLE 5.26. From Example 5.12, we know the cosine function is analytic on all of $\mathbb{R}$. Hence the function $f(x) = \cos(x^3)$ is also analytic on all of $\mathbb{R}$. To get its Maclaurin series, we substitute x^3 for x in Display 5.6, obtaining

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{6n}.$$

Note how much easier that was than computing successive derivatives of f at zero. In fact these can be read off from the last display. For example, the coefficient of x^6 is $\frac{-1}{2}$. But by definition, we know the coefficient of x^6 in the Maclaurin series of f must be $\frac{f^{(6)}(0)}{6!}$. Equating these two expressions, we obtain $f^{(6)}(0) = \frac{6!}{-2} = -360$. (If you have time to waste, try getting that by direct differentiation.)

EXAMPLE 5.27. Substituting $-x^2$ for x in Equation 5.7, see that the Maclaurin series for $\frac{1}{1+x^2}$ converges to $\frac{1}{1+x^2}$ throughout the open interval $(-1, 1)$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n}, \qquad -1 < x < 1.$$

From here, integration gives us the Maclaurin series for arctan.

$$\arctan x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}, \qquad -1 < x < 1.$$

Exercises

Proofs and Computations

PROBLEM 5.1. Carefully state and independently prove the special case of Taylor's Theorem 5.2 corresponding to $n = 1$.

PROBLEM 5.2. Write out details of the inductive proof needed to establish Part (1) of Taylor's Theorem.

PROBLEM 5.3. Find Maclaurin series for the following functions:

(1) $5x^3 - 4x + 13$
(2) $(1-x)^4$
(3) $\sin x$
(4) $\cos 2x$
(5) $\sinh x = \frac{e^x - e^{-x}}{2}$
(6) $\frac{1}{1+3x}$
(7) $\ln(1+2x)$
(8) $\sqrt{1+x}$.

PROBLEM 5.4. Use appropriate third degree Taylor polynomials to approximate each of the following. Try to choose the function f and the base point a as efficiently as possible.

(1) $\sin .01$,
(2) $\sin 1$,
(3) e,
(4) $\sqrt{3.9}$,

PROBLEM 5.5. Give reasonable upper bounds on the errors of your approximations in the preceding problem.

PROBLEM 5.6. Outline efficient methods for approximating each of the following to five decimal place accuracy. In each part, specify the relevant function f, base point a, and number of terms n.

(1) $\cos .01$,
(2) $\sin 1$,
(3) $\sqrt{99}$.

PROBLEM 5.7. Find an *exact* expression for the remainder r_n in the Maclaurin series of $\frac{1}{1-x}$.

PROBLEM 5.8. Find the coefficient of x^5 in the expansion of $(x-3)^{10}$.

PROBLEM 5.9. Find the partial fraction expansion of $\frac{x^3+5x^2}{(x-2)^4}$. Hint: Express the numerator in terms of powers of $x-2$.

PROBLEM 5.10. A theorem of Abel states that if a numerical series $\sum b_n$ has a real sum L, then $\lim_{x\to 1^-} \sum b_n x^n = L$ as well. Use this result to obtain a series representation of π. Hint: What is $\arctan 1$?

PROBLEM 5.11. Suppose the numerical sequence $(b_n)_{n=0}^{\infty}$ decreases monotonically to zero. Show that the error involved in approximating the alternating series $\sum_{k=0}^{\infty}(-1)^k b_k$ by its n'th partial sum is at most b_{n+1}.

PROBLEM 5.12. How many terms from the series expansion you found in Problem 14 are needed to approximate π to within .001 ?

PROBLEM 5.13. Show that the exponential function is analytic on $\mathbb{R}$.

PROBLEM 5.14. Use a Taylor series centered at 0 prove that the function $f(x) = \frac{1}{x}$ is analytic on the interval $(0, 2)$.

Operations on Taylor Polynomials and Series

PROBLEM 5.15. Find Maclaurin series for the following functions:

(1) $x^3 \cos(x^4)$,
(2) $\frac{1}{1+x^4}$,
(3) $\arctan x^2$,
(4) $\int_0^x \cos t^2 dt$.

PROBLEM 5.16. Find closed form expressions for the following power series by manipulating Equation 5.7.

(1) $\sum_{n=0}^{\infty} 3^n (x-4)^n$
(2) $\frac{x^n}{2^n}$
(3) $\sum_{n=0}^{\infty} n x^{n-1}$
(4) $\sum_{n=0}^{\infty} n x^n$
(5) $\sum_{n=0}^{\infty} n^2 x^n$
(6) $\sum_{n=1}^{\infty} \frac{x^n}{n}$
(7) $\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

PROBLEM 5.17. Find the Maclaurin series of $f(x) = \exp(x^2)$, and use it to evaluate $f^{(10)}(0)$.

PROBLEM 5.18. Suppose p is a polynomial of degree at most n satisfying $\lim_{x \to 0} \dfrac{p(x)}{x^n} = 0$. Argue inductively on n to show that $p \equiv 0$. This fact was used in the proof of Proposition 5.3.

PROBLEM 5.19. State and prove an analogue of Corollary 5.15 to go along with Proposition 5.23.

PROBLEM 5.20. State and prove an analogue of Corollary 5.15 to go along with Proposition 5.25.

PROBLEM 5.21. Suppose f is has derivatives of all orders in some interval $(-r, r)$, and c is a non-zero real number. State and prove analogues of Proposition 5.14 and Corollary 5.15 relating the Maclaurin data associated with the function cf to the corresponding data for the function f.

PROBLEM 5.22. Suppose f is has derivatives of all orders in some interval $(-r, r)$, while g has derivatives of all orders in some interval $(-s, s)$. State and prove analogues of Proposition 5.14 and Corollary 5.15 relating the Maclaurin data associated with the function $f + g$ to the corresponding data for the individual functions f.

CHAPTER 6

Power Series

The exponential function is one of the most important functions in mathematics. If you look back at your precalculus/calculus text however, you will realize that it took quite a bit of effort to define quantities like e and $e^{\sqrt{2}}$. On the other hand the Maclaurin series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \tag{6.1}$$

of exp is rather simple. It is tempting to use this expression to *define* the exponential function. The steps to making this precise include

(1) Show that for each $x \in \mathbb{R}$, the numerical series determined by (6.1) converges to some number $S(x)$.
(2) Show that the function S is continuous.
(3) Show that it makes sense to differentiate (6.1) term by term.
(4) Close the cycle by showing that (6.1) is the Maclaurin series of S.

It then follows easily that S is its own derivative and enjoys other properties we expect of the exponential function.

The purpose of this Chapter is to carry out the above program for general power series $\sum b_n x^n$. This is important in differential equations, where it is frequently easy to find series solutions of initial value problems even when such solutions cannot be expressed in terms of elementary functions.

We begin with an official record of the relevant definitions.

Definition 6.1. Suppose A is a non–empty set of real numbers and for each natural number n, we are given a function $f_n : A \to \mathbb{R}$.

(1) (f_n) is said to be a *sequence of functions on A*.
(2) The *series* $\sum f_n$ is comprised of two function sequences: the *sequence of terms* (f_n) and its *sequence of partial sums* (s_n). The latter are defined by $s_n := f_1 + \dots f_n$.
(3) The *domain of convergence* of $\sum f_n$ is the set D of real numbers x for which the numerical series $\sum f_n(x)$ converges.
(4) The *sum* of the series $\sum f_n$ is the function S whose value at each $x \in D$ is the sum of the numerical series $\sum f_n(x)$.

1. Domains of Convergence

A *power series* is a function series of the form $\sum b_n(x-a)^n$ where a is a real number and (b_n) is a sequence of real numbers. We say this series is *centered* at a. We can use the tests of Chapter 3 to determine the domain of convergence of a power series. The ratio test is particularly effective since it can be applied to most x simultaneously.

EXAMPLE 6.2. Determine the domains of convergence of the following power series.

(1) $\sum \frac{x^n}{2^n}$
(2) $\sum \frac{x^n}{n2^n}$
(3) $\sum \frac{x^n}{n^2 2^n}$
(4) $\sum \frac{(x-2)^n}{n2^n}$
(5) $\sum n! x^n$
(6) $\sum \frac{x^n}{n!}$

SOLUTION. (1) We apply the absolute version of the ratio test, Corollary 2.8. (Technically, the following computation makes no sense when $x = 0$, but convergence is obvious in that case.)

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{2^n x^{n+1}}{2^{n+1} x^n} \right| = \frac{|x|}{2}.$$

Thus we get convergence if $|x| < 2$, divergence when $|x| > 2$, and no information when $|x| = 2$. Testing $x = 2, -2$ individually, we get the numerical series $\sum 1$ and $\sum (-1)^n$ respectively, both of which diverge by the n'th term test.

Thus the domain of convergence is the open interval $(-2, 2)$.

(2) The ratio test gives us the same information as in Part (1):

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)2^n x^{n+1}}{n2^{n+1} x^n} \right| = \frac{|x|}{2}.$$

Thus we get convergence if $|x| < 2$, divergence when $|x| > 2$, and no information when $|x| = 2$. Testing $x = 2, -2$ individually, we get the numerical series $\sum \frac{1}{n}$ and $\sum \frac{(-1)^n}{n}$ respectively; the former (harmonic) series diverges while the latter series converges by the alternating series test.

Thus the domain of convergence is the interval $[-2, 2)$.

(3) Once again, the ratio test gives convergence if $|x| < 2$, divergence when $|x| > 2$, and no information when $|x| = 2$. Testing $x = 2, -2$ individually, we get the numerical series $\sum \frac{1}{n^2}$ and $\sum \frac{(-1)^n}{n^2}$, both of which converge absolutely.

Thus the domain of convergence is the closed interval $[-2, 2]$.

(4) One could repeat the *computations* of Part (2), but it easier to note that the substitution $y = x-2$ transforms this series to $\sum \frac{y^n}{n2^n}$. The *result* of Part (2) shows that the transformed series converges for

$-2 \le y < 2$ whence $\sum \frac{(x-2)^n}{n2^n}$ converges for $-2 \le x-2 < 2$, i.e., for $0 \le x < 4$.

Thus the domain of convergence is the half–open interval $[0, 4)$.

(5) The series converges for $x = 0$. For any other x, we have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)!x^{n+1}}{n!x^n}\right| = \lim_{n\to\infty} (n+1)|x| = \infty.$$

Thus the domain of convergence is the singleton set $\{0\}$.

(6) As in Part (5), the result of the ratio–test computation does not depend on x.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{n!x^{n+1}}{(n+1)!x^n}\right| = \lim_{n\to\infty} \frac{|x|}{(n+1)} = 0.$$

Thus the domain of convergence is $\mathbb{R}$.

□

The following proposition shows that it is no accident that all the answers to the preceding exercise are intervals.

PROPOSITION 6.3. *Suppose the power series* $\sum b_n x^n$ *converges at* $x = c$ *and that* $0 < s < |c|$. *Then there are constants* $K > 0$ *and* $\rho \in (0, 1)$ *such that* $|b_n x^n| \le K\rho^n$ *whenever* $0 \le |x| \le s$.

In particular the power series $\sum b_n x^n$ *converges whenever* $|x| < c$.

PROOF. Since the terms of every convergent series are bounded, there is some number K such that $|b_n c^n| \le K$ for all n. Set $\rho := \frac{s}{|c|}$. Then $0 < \rho < 1$. Moveover, for $|x| \le s$, we have $|b_n x^n| \le K\rho^n$ for all n. Thus $\sum b_n x^n$ converges absolutely by comparison with the geometric series $\sum K\rho^n$. □

COROLLARY 6.4. *The domain of convergence of any power series centered at* a *is an interval centered at* a. *It may, however, include neither, one, or both of the endpoints of the interval.*

PROOF. The assertion concerning endpoints is settled by Example 6.2. Power series centered at a real number a can be transformed to power series centered at 0 by making the substitution $y = x - a$.

Consider then a power series $\sum b_n x^n$ centered at 0. If the domain of convergence of this series is bounded, it has a least upper bound r. Proposition 6.3 then shows that the series converges whenever $|x| < r$; the *contrapositive* of Proposition 6.3 shows that the series diverges whenever $|x| > r$. The possible sets of convergence are thus $(-r, r), [-r, r), (-r, r]$, and $[-r, r]$, all of which are intervals centered at 0. (When $r = 0$, only the interval $[r, r]$ which reduces to the singleton set $\{0\}$ is possible.)

If the domain of convergence of $\sum b_n x^n$ is unbounded, Proposition 6.3 shows that it converges for all real x. This completes the proof since $\mathbb{R}$ is considered to be an interval (centered at any point). □

Definition 6.5. The domain of convergence of a power series is also called its *interval* of convergence. If this interval is bounded, half its length is referred to as the *radius of convergence* of the power series. A power series which converges for all x is said to have an *infinite radius of convergence.*

The following corollary of the ratio test could be used to abbreviate the computations we made in Example 6.2. The proof is left as an exercise.

Proposition 6.6. *If* $\lim \left|\frac{b_n}{b_{n+1}}\right| = r$ *exists as a finite number or is infinite, then* r *is the radius of convergence of the power series* $\sum b_n(x-a)^n$*.*

2. Uniform Convergence

It is the goal of this chapter to show that sums of power series are 'nice" functions in the following sense.

Theorem 6.7. *Suppose* $\sum b_n(x-a)^n$ *is a power series whose radius of convergence* r *is strictly positive, and write* J *for the open interval* $(a-r, a+r)$*. Define a function* S *by setting* $S(x_0) = \sum b_n(x_0-a)^n$ *for each* $x_0 \in J$*. Then* S *has derivatives of all orders throughout the interval* J*, and the original power series is the Taylor series of* S *about* a*. In particular,* S *is analytic throughout* J*.*

Our first job is showing that S is continuous.

Definition 6.8. Let D be a non-empty set of real numbers. A sequence (f_n) of functions *converges pointwise* to function L on D if $\lim_{n\to\infty} f_n(x) = L(x)$ for each $x \in D$.

The adverb "pointwise" is used to emphasize the point of view that we are considering one x at a time. Reviewing Definition 2.4, this means given $x \in D$ and $\epsilon > 0$, there is a natural number N **allowed to depend on** x **as well as** ϵ such that $|f_n(x) - L(x)| < \epsilon$ whenever $n \geq N$. The drawback to this concept is that pointwise limits of nice functions may be badly behaved.

Example 6.9. Each of the power functions $f_n(x) = x^n$ is continuous everywhere, but their pointwise limit on $[0,1]$ is the discontinuous function $L(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}$.

This is a serious setback to our proof of Theorem 6.7. By definition, the sum of a power series is the *pointwise* limit of its partial sums. Those partial sums seem as nice as could be (polynomials), but the example suggests that all could be lost when we take pointwise limits. We get around this problem by introducing a stronger notion of convergence.

Definition 6.10. Let D be a non-empty set of real numbers. A sequence (f_n) of functions *converges uniformly* to function L on D if given $\epsilon > 0$, there is a natural number N such that $|f_n(x) - L(x)| < \epsilon$ simultaneously for all $x \in D$ and all $n \geq N$.

The key point here is that we have to choose N without knowing what x we're talking about.

Example 6.11. Define functions f_n and g_n on $\mathbb{R}$ by $f_n(x) = \frac{1}{n}\sin x$ and $g_n(x) = \frac{x}{n}$. For each individual x, we have $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} g_n(x) = 0$, so both sequences of functions converge to zero pointwise on $\mathbb{R}$.

(f_n) also converges to zero uniformly on $\mathbb{R}$. Indeed, given $\epsilon > 0$, choose a natural number $N > \frac{1}{\epsilon}$. Then for all $n \geq N$ and all $x \in \mathbb{R}$, we have

$$|f_n(x) - 0| = |\frac{1}{n}\sin x| \leq \frac{1}{n} \leq \frac{1}{N} < \epsilon,$$

as required.

On the other hand (g_n) does not converge (to zero) uniformly. To see this, take $\epsilon = \frac{1}{2}$ and suppose N is some natural number. Then, taking $x = N = n$, we have $|f_n(x) - 0| = 1 > \epsilon$, so there is no N which simultaneously works for this ϵ and all x.

We now show that uniform convergence does preserve continuity.

Proposition 6.12. *Suppose (f_n) is a sequence of functions converging uniformly to a function L on an interval $D \subset \mathbb{R}$ and $a \in D$. If each f_n is continuous at a, then L is continuous at a as well.*

Proof. For each $x \in D$ and $N \in \mathbb{N}$, the triangle inequality yields

$$|L(x) - L(a)| \leq |L(x) - f_N(x)| + |f_N(x) - f_N(a)| + |f_N(a) - L(a)|.$$

Given $\epsilon > 0$, apply uniform convergence to find $N \in \mathbb{N}$ so that $|f_n(x) - L(x)| < \frac{\epsilon}{3}$ for all x (including a) in D. Then apply the continuity of f_N to get $\delta > 0$ so that $|f_N(x) - f_N(a)| < \frac{\epsilon}{3}$ whenever $|x - a| < \delta$. Thus when $|x - a| < \delta$, we can continue the preceding display to get

$$|L(x) - L(a)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,$$

and we have satisfied the continuity Definition 4.7. □

In particular, the convergence in Example 6.9 cannot be uniform.

While power series need not converge uniformly on their domains of covergence, they do converge uniformly on smaller intervals. That will set us up to apply Proposition 6.12. We need one more definition to carry out this program.

Definition 6.13. A sequence (f_n) of functions is *uniformly cauchy* on a domain D if for each $\epsilon > 0$, there is a natural number N such that

$$|f_m(x) - f_n(x)| < \epsilon \quad \text{for all } m, n \geq N \text{ and all } x \in D \tag{6.2}$$

Proposition 6.14. *Let D be a non-empty subset of $\mathbb{R}$. Every uniformly cauchy sequence on D is uniformly convergent on D.*

PROOF. Suppose (f_n) is uniformly convergent on D. For each fixed $x \in D$, $(f_n(x))$ is a cauchy *numerical* sequence and hence converges to some number $L(x)$. In other words, (f_n) converges pointwise to L, and we must show this convergence is uniform.

So let $\epsilon > 0$ be given and choose N as in Display 6.2 corresponding to $\frac{\epsilon}{2}$. Now suppose $x \in D$ and $n \geq N$. Then for each $m \geq N$, the triangle inequality yields

$$|f_n(x) - L(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - L(x)| \leq \frac{\epsilon}{2} + |f_m(x) - L|$$

Now choose m (depending on x) so large that $|f_m(x) - L| < \frac{\epsilon}{2}$. (It's okay to have m depend on x becuase m doesn't appear on the lefthand side of the Display). Thus the Display continues to show $|f_n(x) - L| < \epsilon$ as desired. □

DEFINITION 6.15. Let D be a subset or $\mathbb{R}$. An infinite series $\sum f_n$ of functions *converges uniformly* to a function S on D if its sequence (s_n) of partial sums of the series converges uniformly to S.

THEOREM 6.16. *(Weierstrauss M-test) Suppose $\sum f_n$ is a series of functions on a domain D, and $\sum M_n$ is a convergent series of positive numbers. If $|f_n(x)| \leq M_n$ for all n and all $x \in D$, then the function series $\sum f_n$ converges uniformly on D.*

PROOF. Write (s_n) for the sequence of partial sums of (f_n). Given $\epsilon > 0$, choose N so that $\sum_{k=m}^{n} M_k < \epsilon$ whenever $n \geq m \geq N$. Then for all $n \geq m \geq M$ and all $x \in D$, we have

$$|s_n(x) - s_m(x)| = \left| \sum_{k=m}^{n} f_k(x) \right| \leq \sum_{k=m}^{n} M_n < \epsilon$$

This shows the sequence (s_n) is uniformly Cauchy, hence uniformly convergent by Proposition 6.14. □

We're now ready to put everything together. For ease of reading, we restrict attention to power series centered at 0.

COROLLARY 6.17. *Suppose the power series $\sum b_n x^n$ has radius of convergence $r > 0$, and let s be a positive number which is less than r. Then the power series converges uniformly on the interval $|x| \leq s$*

PROOF. Fix c in the open interval (s, r) and apply Proposition 6.3 to get $K > 0$ and $\rho \in (0, 1)$ such that $|b_n x^n| \leq K\rho^n$ for all n and all $|x| \leq s$. The proof is completed by applying the Weirstrauss M-test with $M_n = K\rho^n$. □

Here is the first step towards Theorem 6.7.

COROLLARY 6.18. *Suppose the power series $\sum b_n x^n$ has radius of convergence $r > 0$ and denote the function which is the limit of its partial sums by S. Then S is continuous on the open interval $|x| < r$.*

PROOF. Fix $s < r$. Each partial sum s_n of the power series is a polynomial and hence continuous everywhere. The previous Corollary tells us (s_n) converges uniformly on $[-s, s]$, whence Proposition 6.12 yields continuity of S on each closed subinterval of $|x| < r$. That's enough to give continuity of S throughout $|x| < r$. □

3. Analyticity of Power Series

In this section, we complete the proof of Theorem 6.7. The following Proposition means that integration preserves uniform convergence.

PROPOSITION 6.19. *Suppose (f_n) is a sequence of continuous functions converging uniformly to a function g on the interval $|x| < r$. Define functions F_n, G by*

$$F_n(x) = \int_0^x f_n, \quad G(x) = \int_0^x g, \quad \text{for all } x \in (-r, r).$$

Then (F_n) converges uniformly to G on $(-r, r)$.

PROOF. Given $\epsilon > 0$, find a natural number N so that $|f_n(x)-g(x)| < \frac{\epsilon}{r}$ for all $n \geq N$ and for all $|x| < r$. Then whenever $n \geq N$ and $|x| < r$, we have

$$|F_n(x) - G(x)| = \left| \int_a^x (f_n - g) \right| \leq |x - a| \frac{\epsilon}{r} < \epsilon$$

□

Unfortunately, differentiation does not preserve uniform convergence.

EXAMPLE 6.20. Take $f_n(x) = \frac{1}{n} \sin n^2 x$. Then (f_n) converges to 0 uniformly on $\mathbb{R}$. However $f_n'(0) = n$ for each n so (f_n') does not even converge pointwise.

The next Proposition overcomes this last obstacle. A function f is said to be *continuously differentiable* if its derivative f' is continuous.

PROPOSITION 6.21. *Let (f_n) be a sequence of continuously differentiable functions on the open interval $(-r, r)$. Suppose (f_n) converge uniformly to some function g on $(-r, r)$ while (f_n') converge uniformly to some function h on that interval. Then $h = g'$ on $(-r, r)$.*

PROOF. Applying Proposition 6.19 at the last step, we have

$$g(x) - g(0) = \lim_{n\to\infty} f_n(x) - f_n(0) = \lim_{n\to\infty} \int_0^x f_n' = \int_0^x h, \quad |x| < r.$$

Differentiating with respect to x gives the desired result. □

PROPOSITION 6.22. *Suppose the power series $\sum_{n=0}^{\infty} b_n x^n$ has radius of convergence $r > 0$ and denote the function which is the limit of its partial sums by S.*

(1) *The derived power series $\sum_{n=1}^{\infty} n b_n x^{n-1}$ converges on $(-r, r)$.*
(2) *S is differentiable on the the interval $(-r, r)$.*

(3) *S' is the sum of the derived power series on $(-r, r)$.*

PROOF. Fix numbers c, s satisfying $0 < s < c < r$ and apply Proposition 6.3 to get $K > 0$ and $\rho \in (0, 1)$ such that $|b_n x^n| \leq K\rho^n$ for all n and all $|x| \leq s$. Since the numerical series $\sum Kn\rho^{n-1}$ by the ratio test, the Weierstrauss M-test tells us that the derived power series converges uniformly on the interval $|x| \leq s$. This means the partial sum sequence (s'_n) of the derived series converges uniformly to some function T on $|x| \leq s$. Since the original power series also converges uniformly on that interval, its sequence (s_n) of partial sums converges uniformly to S on that interval as well. Now Proposition 6.21 guarantees that $S' = T$ which simultaneously establishes (2) and (3) on the interval $(-s, s)$. Since this is true for every $s < r$, the proof is complete. □

We're now ready to prove the main result of this Chapter, which we restate for convenience.

THEOREM 6.23. *Suppose $\sum b_n(x-a)^n$ is a power series whose radius of convergence r is strictly positive, and write J for the open interval $(a - r, a + r)$. Define a function S by setting $S(x_0) = \sum b_n(x_0 - a)^n$ for each $x_0 \in J$. Then S has derivatives of all orders throughout the interval J, and the original power series is the Taylor series of S about a. In particular, S is analytic throughout J.*

PROOF. For notational simplicity, we continue to assume $a = 0$.

We have just shown the derived power series converges to S' on $(-r, r)$. This can be repeated inductively to get

$$S^{(k)}(x) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} b_n x^{n-k}, \quad k \in \mathbb{N},\ |x| < r. \tag{6.3}$$

This establishes the assertion that S has derivatives of all orders on J. Plugging in $x = 0$ in the display, we conclude $S^{(k)}(0) = k!b_k$ for each natural number k, that is, the b_k's are the Maclaurin coefficients of S and $\sum b_n x^n$ is indeed the Taylor series of S. Analyticity of S now follows since the original power series converges to it on $(-r, r)$ by assumption. □

Exercises

Domains of Convergence

PROBLEM 6.1. Determine the domains of convergence of the following power series:

(1) the series $\sum \frac{x^n}{3^n\sqrt{n}}$,
(2) the series obtained via term by term differentiation of the series from Part (1)
(3) the series obtained via term by term integration of the series from Part (1)

(4) the series $\sum \frac{(x-2)^{2n}}{\sqrt{n}}$.

PROBLEM 6.2. Prove Proposition 6.6.

PROBLEM 6.3. Use a comparison argument like that of Proposition 6.3 to show that the power series $\sum b_n x^n$ and $\sum n b_n x^n$ have the same radius of convergence.

Uniform Convergence

PROBLEM 6.4. Prove that the sequence (x^n) of power functions does not converge to 0 uniformly on the open interval $(0,1)$.

PROBLEM 6.5. Suppose $0 \leq r < 1$. Prove that the sequence (x^n) of power functions does converge to 0 uniformly on the interval $[-r, r]$.

PROBLEM 6.6. Give an example of a sequence (f_n) of bounded functions which converge pointwise to an unbounded function f on the interval $(0,1)$.

PROBLEM 6.7. Let (f_n) be a sequence of bounded functions which converge uniformly to a function f on some interval J. Prove that f is also bounded.

PROBLEM 6.8. Prove that the power series $\sum \frac{1}{n^2} x^n$ converges uniformly on the interval $[-1,1]$.

PROBLEM 6.9. Suppose the infinite series of functions $\sum f_n$ converges uniformly on some interval J. Prove that its sequence (f_n) of terms must converge to 0 uniformly on J.

PROBLEM 6.10. Suppose the power series $\sum b_n x^n$ converges uniformly on $\mathbb{R}$. Prove there is some natural number N such that $b_n = 0$ for all $n \geq N$. In other words, such a series must be a polynomial.

PROBLEM 6.11. Prove that the power series $\sum x^n$ does not converge uniformly on the interval $(0,1)$.

PROBLEM 6.12. Prove that the power series $\sum \frac{1}{n} x^n$ does not converge uniformly on the interval $(0,1)$.

PROBLEM 6.13. Prove that the power series $\sum \frac{1}{n} x^n$ does converge uniformly on the interval $(-1,0)$.

PROBLEM 6.14. Establish Equation 6.3.

PROBLEM 6.15. Use Equation 6.3 to show that if the two power series $\sum b_n (x-a)^n$ and $\sum c_n (x-a)^n$ converge to the same function in some open interval about a, then $b_n = c_n$ for all n.

CHAPTER 7

Complex Sequences and Series

1. Motivation

In Chapter 1, we characterized the real number system as a complete ordered field.

PROPOSITION 7.1. *The equation $x^2 = -1$ does not have a solution in any ordered field.*

PROOF. Let x be an element of an ordered field F. By Proposition 1.8, we know that $x^2 + 1 \in F^+$, whence $x^2 + 1 \neq 0$ by trichotomy. $\square$

This is somewhat of a nuisance in various areas of mathematics. For example, because $x^2 - 1$ can be factored, the method of partial fractions yields $\dfrac{2}{x^2-1} = \dfrac{1}{x-1} - \dfrac{1}{x+1}$, whence

$$\int \frac{2}{x^2-1}\,dx = \ln\frac{x-1}{x+1} + C.$$

On the other hand, (real) partial fractions cannot be used to simplify $\frac{2}{x^2+1}$, and the corresponding integral looks quite different:

$$\int \frac{2}{x^2+1}\,dx = 2\arctan x + C.$$

EXAMPLE 7.2. A similar problem arises in differential equations. Substituting $y = e^{rx}$ in the differential equation

$$y'' - y = 0, \tag{7.1}$$

we obtain $(r^2 - 1)e^{rx} = 0$, which is satified when $r = \pm 1$; this leads to $y = C_1e^x + C_2e^{-x}$ as the general solution to Equation 7.1.

If we try the same substitution in the differential equation

$$y'' + y = 0, \tag{7.2}$$

we obtain $(r^2 + 1)e^{rx} = 0$, which is not satisfied by any real number r; in fact, the general solution to Equation 7.2 is the totally different-looking $y = C_1\cos x + C_2\sin x$.

In seeking to understand Example 7.2, it is natural to try to find a larger field F which contains the real numbers, together with a new number "i" satisfying the equation $i^2 = -1$. Since fields are closed under addition and multiplication, for each $a, b \in \mathbb{R}$, there must be a number $a + bi$ in F.

Moreover, the commutative, associative, and distributive laws dictate how these numbers must be added and multiplied. For $a, b, c, d \in \mathbb{R}$,

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bdi^2 + (ad + bc)i = (ac - bd) + (ad + bc)i.$$

2. Complex Numbers

It is not difficult to make this informal procedure precise. We define $\mathbb{C}$ to be the collection of all ordered pairs of real numbers, equipped with the operations:

$$(a, b) + (c, d) = (a + c, b + d)$$

$$(a, b)(c, d) = (ac - bd, ad + bc)$$

It must be shown that these operations make $\mathbb{C}$ into a field, i.e., that Definition 1.2 is satisfied. Closure and commutativity are obvious; checking associativity and distributivity is tedious but straightforward. The additive identity is $(0, 0)$, while the multiplicative identity is $(1, 0)$. Clearly $(-a, -b)$ provides the additive inverse of (a, b). Finally, if $(a, b) \neq (0, 0)$, a direct computation confirms $(\frac{a}{a^2+b^2}, \frac{-b}{a^2+b^2})$ as its multiplicative inverse.

We call $\mathbb{C}$ the *complex number field.* By introducing i as an abbreviation for $(0, 1)$, and a as an abbreviation for $(a, 0)$, we recover our motivating notation $a + bi = (a, b)$.

You will have to wait for MATH 4150 for a proper appreciation of complex numbers. For example, although we have only built in a root of the polynomial $x^2 + 1$, it turns out that *every* non-constant polynomial has a root in $\mathbb{C}$. The modest goal of the rest of this chapter is simply to adapt our earlier results to the complex setting.

Of course, Proposition 7.1 implies that $\mathbb{C}$ cannot be ordered. This is why absolute values (which are real) come up so often in the study of $\mathbb{C}$; the *triangle inequality* (Proposition 7.4.3) is particularly important.

Definition 7.3. Let $z = a + bi$, with $a, b \in \mathbb{R}$.

(1) a is called the *real part* of z, denoted $\mathrm{Re}\,(z)$
(2) b is called the *imaginary part* of z, denoted $\mathrm{Im}\,(z)$
(3) z is *real* if $b = 0$ and *pure imaginary* if $a = 0$
(4) $\sqrt{a^2 + b^2}$, called the *absolute value* of z, is denoted $|z|$.

Proposition 7.4. *Let $z, w \in \mathbb{C}$.*

(1) $|\,\mathrm{Re}\,(z)\,| \leq |z|$
(2) $|\,\mathrm{Im}\,(z)\,| \leq |z|$
(3) $|z + w| \leq |z| + |w|$

Proof. Write $z = a + bi, w = c + di$ with $a, b, c, d \in \mathbb{R}$. (1) follows by applying non-negative square roots to the inequality $a^2 \leq a^2 + b^2$; the proof of (2) is similar.

For (3), note first that

$$(a^2 + b^2)(c^2 + d^2) - (ac + bd)^2 = (bc - ad)^2.$$

Since the right-hand side of this equation is non-negative, we conclude that

$$ac + bd \leq \sqrt{(a^2 + b^2)(c^2 + d^2)} = |z||w|.$$

It follows that

$$\begin{aligned} |z + w|^2 =& (a + c)^2 + (b + d)^2 = a^2 + b^2 + c^2 + d^2 + 2(ac + bd) \\ \leq& a^2 + b^2 + c^2 + d^2 + 2|z||w| = (|z| + |w|)^2. \end{aligned}$$

Taking non-negative square roots yields the desired result. □

3. Complex Sequences

Definition 7.5. A *sequence of complex numbers* is a function mapping $\mathbb{N}$ into $\mathbb{C}$.

Unless indicated otherwise, all sequences discussed below will be complex sequences. Convergence is defined as in the real case.

Definition 7.6. A complex sequence (a_n) is said to *converge* to the number L if for each number $\epsilon > 0$ there is a natural number N such that $|a_n - L| < \epsilon$ whenever $n \geq N$. In this case the number L is called the *limit of the sequence* (a_n) and we write $\lim_{n\to\infty} a_n = L$. A sequence which does not converge to any complex number is said to *diverge*.

Proposition 7.7. *In order for a complex sequence to converge, it is necessary and sufficient that its real and imaginary parts be convergent.*

Proof. Suppose (z_n) is a complex sequence converging to a number L, and let $\epsilon > 0$ be given. Choose N such that $n \geq N$ implies $|z_n - L| < \epsilon$. In view of Proposition 7.4, we also have $|\operatorname{Re}(z_n) - \operatorname{Re}(L)| < \epsilon$ and $|\operatorname{Im}(z_n) - \operatorname{Im}(L)| < \epsilon$ whenever $n \geq N$. This means $(\operatorname{Re}(z_n))$ and $(\operatorname{Im}(z_n))$ converge to $\operatorname{Re}(L)$ and $\operatorname{Im}(L)$ respectively.

Conversely, suppose $\lim_{n\to\infty} \operatorname{Re}(z_n) = a$, and $\lim_{n\to\infty} \operatorname{Im}(z_n) = b$. Given $\epsilon > 0$, choose N so that $n \geq N$ implies

$$|(\operatorname{Re}(z_n) - a)| < \frac{\epsilon}{2} \quad \text{and} \quad |(\operatorname{Im}(z_n) - b)i| < \frac{\epsilon}{2}.$$

Then whenever $n \geq N$, the triangle inequality yields

$$|z_n - (a + bi)| \leq |(\operatorname{Re}(z_n) - a)| + |(\operatorname{Im}(z_n) - b)i| < \epsilon,$$

whence $\lim_{n\to\infty} z_n = a + bi$. □

The absence of an order on $\mathbb{C}$ makes it impossible to talk about upper or lower bounds of complex sequences. However, we can use absolute values to define boundedness.

Definition 7.8. A complex sequence (a_n) is *bounded* if there is a positive (real) number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

PROPOSITION 7.9. *Every convergent complex sequence is bounded.*

PROOF. Suppose (a_n) converges to L. Taking $\epsilon = 1$ (any *fixed* ϵ would do), we find a natural number N such that $n \geq N$ implies $|a_n - L| < 1$. But $|a_n| \leq |a_n - L| + |L|$ by the triangle inequality. Thus $|a_n| < |L| + 1$ for all $n \geq N$. The proof is completed by taking $M = \max\{|a_1|, |a_2|, \ldots, |a_{N-1}|, |L| + 1\}$. □

Most of the remaining results in this chapter parallel our work in Chapters 2, 3, and 5. Proofs can be constructed either by mimicking the earlier arguments or by combining Proposition 7.7 with the earlier *results.*

PROPOSITION 7.10. *If (z_n) and (w_n) are convergent complex sequences, then*

(1) $\lim(z_n + w_n) = \lim z_n + \lim w_n$*;*
(2) $\lim z_n w_n = \lim z_n \lim w_n$*;*
(3) *If* $\lim w_n \neq 0$ *then* $\lim \frac{z_n}{w_n} = \frac{\lim z_n}{\lim w_n}$.

DEFINITION 7.11. A sequence (a_n) of complex numbers is *Cauchy* if for each $\epsilon > 0$ there is a natural number N such that $|a_m - a_n| < \epsilon$ for all natural numbers m and n greater than or equal to N.

PROPOSITION 7.12. (1) *Every complex sequence which converges is Cauchy.*
(2) *Every Cauchy sequence of complex numbers has a complex limit.*

4. Complex Series

DEFINITION 7.13. Let $(a_n)_{n \geq N}$ be a sequence of complex numbers. Define a new sequence $(s_n)_{n \geq N}$ inductively by setting $s_N = a_N$ and $s_{n+1} = s_n + a_{n+1}$ for $n \geq N$. We call s_n the *n'th partial sum* of the original sequence; it is also denoted by $\sum_{i=N}^{n} a_i$.

DEFINITION 7.14. An *(infinite) series* is a pair of sequences $(a_n)_{n \geq N}$ and $(s_n)_{n \geq N}$ where the latter sequence is the sequence of partial sums of the former sequence. It is denoted by $\sum_{n=N}^{\infty} a_n$. We call a_n the n'th *term* of the series and we call s_n the n'th *partial sum* of the series.

DEFINITION 7.15. A complex series $\sum_{n=N}^{\infty} a_n$ *converges* if its sequence $(s_n)_{n \geq N}$ of partial sums converges to some complex number S. In this case, S is referred to as the *sum* of the series and we say that the series *converges to* S; we even write $S = \sum_{n=N}^{\infty} a_n$. A series which does not converge is said to *diverge.*

As in the real case, the important thing to keep in mind is that convergence of a series refers to convergence of its sequence of *partial sums.* The notation $\sum_{n=N}^{\infty} a_n$ is ambiguous; one must consider the context to decide whether it stands for a series (i.e., a pair of sequences) or a number (i.e., the sum of a convergent series).

PROPOSITION 7.16. *In order for a complex series to converge, it is necessary and sufficient that its real and imaginary parts be convergent.*

PROPOSITION 7.17 (n'th Term Test). *In order for a complex series to converge, it is necessary that its terms approach zero.*

PROPOSITION 7.18. *Suppose the complex series* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *converge to* A *and* B *respectively and* K *is a complex number.*

(1) *The series* $\sum_{n=1}^{\infty}(a_n + b_n)$ *converges to* $A+B$.
(2) *The series* $\sum_{n=1}^{\infty} Ka_n$ *converges to* KA.

COROLLARY 7.19. *Omitting or changing the first few terms of a series does not affect its convergence.*

DEFINITION 7.20. The complex series $\sum a_n$ is *absolutely convergent* if the series $\sum |a_n|$ is convergent.

PROPOSITION 7.21 (Absolute Convergence Test). *Each absolutely convergent complex series is convergent.*

COROLLARY 7.22 (Ratio Test). *Suppose* $\sum_{n=1}^{\infty} a_n$ *is a complex series for which* $\lim |\frac{a_{n+1}}{a_n}| = L$ *exists as a finite number or is infinite.*

(1) *If* $L < 1$, *then the series converges.*
(2) *If* $L > 1$, *then the series diverges.*

THEOREM 7.23 (Dirichlet's test for convergence). *Let* (z_n) *and* (b_n) *be sequences of numbers, and set* $s_n = z_1 + \cdots + z_n$. *Suppose that* (s_n) *is bounded, while* (b_n) *is a* real *sequence which decreases montonically to zero. Then* $\sum_{n=1}^{\infty} b_n z_n$ *converges.*

EXAMPLE 7.24. Suppose $z \in \mathbb{C}$ has absolute value one, but $z \neq 1$. Then the series $\sum_{n=1}^{\infty} \frac{z^n}{n}$ converges.

PROOF. Apply Dirichlet's Test with $a_n = z^n$ and $b_n = \frac{1}{n}$. The boundedness hypothesis is satisfied since

$$|s_n| = |z + z^2 + z^3 + \cdots + z^n| = \left|\frac{z - z^{n+1}}{1 - z}\right| \leq \frac{2}{|1 - z|}.$$

□

5. Complex Power Series

DEFINITION 7.25. Suppose D is a non–empty set of complex numbers and for each natural number n, we are given a function $f_n : D \to \mathbb{C}$.

(1) (f_n) is said to be a *sequence of functions on* D.
(2) The *sequence of partial sums* (s_n) and the *series* $\sum f_n$ are defined as in the numerical case.
(3) The *domain of convergence* of $\sum f_n$ is the set of complex numbers z for which the numerical series $\sum f_n(z)$ converges.

(4) A *complex power series* is a series of the form $\sum b_n(z-a)^n$ where a is a complex number and (b_n) is a sequence of complex numbers. The power series is said to be *centered* at a.

PROPOSITION 7.26. *Suppose the power series $\sum b_n z^n$ converges when $z = c$. Then the power series $\sum b_n z^n$ converges whenever $|z| < |c|$.*

DEFINITION 7.27. Let D be the domain of convergence of a complex power series centered at a. Then the least upper bound of $\{|z-a| : z \in D\}$ is called the *radius of convergence* of the power series. (If D is unbounded, we say *the radius of convergence is infinite*.)

COROLLARY 7.28. *Suppose $\sum b_n z^n$ has radius of convergence r. If r is infinite, then the series converges for all $z \in \mathbb{C}$. Otherwise, the series converges whenever $|z| < r$ and diverges whenever $|z| > r$. The domain of convergence may include none, all, or some of the points on the circle $|z| = r$.*

PROPOSITION 7.29. *If* $\lim \left|\frac{b_n}{b_{n+1}}\right| = r$ *exists as a finite number or is infinite, then r is the radius of convergence of the power series $\sum b_n(x-a)^n$.*

EXAMPLE 7.30. Example 7.11 shows the domain of convergence of $\sum \frac{z^n}{n}$ is $\{z \in \mathbb{C} : |z| \leq 1, z \neq 1\}$.

Derivatives of functions mapping $\mathbb{C}$ to $\mathbb{C}$ are defined as in the real case: for $f : \mathbb{C} \to \mathbb{C}$,

$$f'(a) := \lim_{z \to a} \frac{f(z) - f(a)}{z - a},$$

provided the limit exists. Basic differentiation rules also carry over, e.g., $D_z(z^n) = nz^{n-1}$. The accompanying theory, however, holds quite a few surprises. For example, while it is easy to construct a differentiable function $f : \mathbb{R} \to \mathbb{R}$ for which $f''(0)$ fails to exist, differentiable functions from $\mathbb{C}$ to $\mathbb{C}$ automatically have derivatives of all orders. We leave these matters, including proof of the next proposition, for later courses.

PROPOSITION 7.31. *(Term by term differentiation) Let $\sum b_n(z-a)^n$ be a complex power series with radius of convergence $r > 0$ and set*

$$f(z) = \sum b_n(z-a)^n, \qquad |z-a| < r. \tag{7.3}$$

(1) *The* derived *power series $\sum nb_n(z-a)^{n-1}$ also has radius of convergence r.*
(2) *f is differentiable on the interval $|z-a| < r$.*
(3)

$$f'(z) = \sum nb_n(z-a)^{n-1}, \qquad |x-a| < r. \tag{7.4}$$

6. De Moivre's Formula

We close this chapter by applying complex power series to explain Example 7.2. To begin, recall the Maclaurin series for the (real) exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \qquad x \in \mathbb{R}.$$

We simply adopt this expansion as our *definition* for the complex exponential function:

$$\exp(z) := \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots, \qquad z \in \mathbb{C}. \tag{7.5}$$

Proposition 7.29 tells us that this power series converges for all z, and Proposition 7.31 assures us that $\exp(z)$ continues to be its own derivative.

REMARK 7.32. We can use a trick to justify the "exponential" property: $\exp(z + w) = \exp(z)\exp(w)$ for all complex numbers z, w.

PROOF. Temporarily fix $a \in \mathbb{C}$, and set $g(z) = \exp(z)\exp(a - z)$. Applying the chain and product rules, we obtain

$$g'(z) = \exp(z)[-\exp(a - z)] + \exp(z)\exp(a - z) = 0.$$

It follows that g is constant, that is, $g(z) = g(0)$ for all $z \in \mathbb{C}$. Applying this with $a = z + w$ yields $\exp(z)\exp(w) = \exp(z + w)$ as desired. □

In the same way, we adapt the power series expansions of the sine and cosine functions to the complex setting.

$$\sin z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots, \qquad z \in \mathbb{C}. \tag{7.6}$$

$$\cos z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \dots, \qquad z \in \mathbb{C}. \tag{7.7}$$

THEOREM 7.33 (De Moivre). $\exp(ix) = \cos x + i \sin x$ *for all* $x \in \mathbb{R}$.

PROOF. Substitute $z = ix$ in the power series 7.5, 7.6, and 7.7. □

Reviewing the "mysterious" Example 7.2, we find the (complex) solution $y = B_1 \exp(ix) + B_2 \exp(-ix)$ to Equation 6.2, which really does look like the solution to Equation 6.1. Substituting from De Moivre's formula, we get

$$y = (B_1 + B_2)\cos x + i(B_1 - B_2)\sin x,$$

which explains our original solution $y = C_1 \cos x + C_2 \sin x$ to Equation 7.2.

Exercises

In the following exercises, z and w stand for complex numbers.

Complex Numbers

Problem 7.1. Compute the multiplicative inverse of $3 + 4i$.

Problem 7.2. Check that the multiplicative inverse axiom holds in $\mathbb{C}$.

Problem 7.3. Check that the distributive property holds in $\mathbb{C}$.

Problem 7.4. Graph the following equations and inequalities in the cartesian plane:

(1) $\operatorname{Re}(z) = 0$,
(2) $|z| = 1$,
(3) $\operatorname{Im}(z) > 3$,
(4) $|z - i| < 2$.

Problem 7.5. The *complex conjugate* $\overline{z}$ of a complex number z is defined by $\overline{z} := \operatorname{Re}(z) - i \operatorname{Im}(z)$. Prove that $|z + w|^2 = |z|^2 + |w|^2 + 2 \operatorname{Re}(z\overline{w})$ for all $z, w \in \mathbb{C}$.

Problem 7.6. Prove that $\big|\, |z| - |w| \,\big| \le |z - w|$ for all $z, w \in \mathbb{C}$.

Complex Sequences

Problem 7.7. Prove Proposition 7.10 by applying Proposition 7.7 to the corresponding real result Proposition 2.20. Hint: For Part (3), note that $\dfrac{1}{w_n} = \dfrac{\overline{w_n}}{|w_n|^2}$.

Problem 7.8. Prove that a complex sequence (a_n) is bounded if and only if both real sequences $(\operatorname{Re}(a_n))$ and $(\operatorname{Im}(a_n))$ are bounded.

Problem 7.9. Use the result of the last problem to give an alternate proof of Proposition 7.9.

Problem 7.10. Prove Proposition 7.12.

Complex Series

Problem 7.11. Prove Proposition 7.16.

Problem 7.12. Prove Proposition 7.17.

Problem 7.13. Prove Proposition 7.21 and Corollary 7.22.

Complex Power Series

Problem 7.14. Prove Proposition 7.26 and Corollary 7.28.

Problem 7.15. Prove Proposition 7.29.

PROBLEM 7.16. Find the domains of convergence of the following complex power series:

(1) $\sum z^n$
(2) $\sum \frac{nz^n}{2^n}$
(3) $\sum \frac{z^n}{n^2}$
(4) $\sum \frac{(z-5)^n}{\sqrt{n}}$

PROBLEM 7.17. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x|x|$. Prove that f is everywhere differentiable, but that $f''(0)$ does not exist.

PROBLEM 7.18. Let $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = z|z|$. Prove that f is not differentiable at 0.

PROBLEM 7.19. Prove that if the derivative of $f : \mathbb{C} \to \mathbb{C}$ is identically zero, then f is constant.

De Moivre's Formula

PROBLEM 7.20. Take real and imaginary parts of the equation

$$\exp(i(x+y)) = \exp(ix)\exp(iy)$$

to derive the usual identities for $\cos(x+y)$ and $\sin(x+y)$.

PROBLEM 7.21. Apply complex exponentials to solve the initial value problem

$$y'' - 2y' + 5y = 0.$$
$$y(0) = 1, \qquad y'(0) = 7.$$

Use De Moivre's Theorem to express your answer in a form which doesn't involve i.

CHAPTER 8

Constructions of $\mathbb{R}$

In this chapter, we outline three separate (but equivalent) approaches to proving the existence of the real number system, which was formulated in Theorem 1.19 without proof. Each approach is based on the time-honored trick of turning a mathematical question into its own answer; this is the technique we used to construct $\mathbb{C}$ from $\mathbb{R}$ in Chapter 6. The three sections of this chapter devoted to these approaches are independent of one another and can be read in any order.

It is not possible to construct something from nothing in mathematics– one must start with undefined terms and axioms. The most primitive starting point is set theory, with its undefined terms "set" and "element" and its axioms guaranteeing that many sets exist. It is possible to construct $\mathbb{R}$ from these alone, but such a project is well beyond the scope of this course. A more common procedure is to axiomitize the existence of the natural number system, and then successively construct larger number systems from there. One rationale for accepting $\mathbb{N}$ is that its existence is intuitively appealing. In fact, you will learn how to construct $\mathbb{Z}$ and $\mathbb{Q}$ from $\mathbb{N}$ in MATH 4000. For this reason, and because you probably feel comfortable with the rational numbers, the constructions of this chapter only describe how to build $\mathbb{R}$ from $\mathbb{Q}$.

The construction of Section 1 is based on formal decimals, that is, we don't assume such expressions have any prior meaning, but we introduce two operations and an order on them, which turn them into a complete ordered field. This is perhaps the most cumbersome of the three approaches. However, because it mirrors the way most people are first introduced to real numbers, you will probably find it intuitive and familiar.

The Dedekind construction of Section 2 is based on the idea that each real number should be determined by the set of rationals which are smaller than it. This approach is more elegant than using decimals, and it is fairly easy to appreciate why it works, but the technical details are still quite messy and we only outline them.

The Cauchy sequence construction of Section 3 is based on the idea that an individual real number should be determined by the *class* of rational sequences converging to it. This approach has the virtue that it is relatively easy to fill in the details; it is presented last because it assumes familiarity with equivalence relations.

1. Formal Decimals

(added by Valery Alexeev)

A complete rigorous proof of all the axioms is quite delicate and requires a lot more time than we are willing to spend on this in our course. We only give an overview of it.

DEFINITION 8.1. A real number x is a sequence

$$\pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots$$

of digits 0 through 9, which is finite on the left but is infinite on the right. By convention, the sequence can be shortened if all the digits starting from some place are zeros:

$$\pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots \alpha_n 000 \dots = \pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots \alpha_n$$

By definition, the two sequences $\pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots \alpha_n 999 \dots$ and $\pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots (\alpha_n + 1)000 \dots$ represent the same real number. Also, $+0$ and -0 are the same.

Except for ± 0, numbers with "+" in front are called positive, and those with "−" in front are called negative.

DEFINITION 8.2. For a real number x as before, the *n-th decimal cut* $x^{(n)}$ is the real number $\pm a_m a_{m-1} \dots a_0.\alpha_1\alpha_2 \dots \alpha_n$ which corresponds to the expression

$$\pm(a_m 10^m + a_{m-1}10^{m-1} + \dots a_0 1 + \alpha_1 10^{-1} + \alpha_2 10^{-2} \dots \alpha_n 10^{-n})$$

In particular, we see that the cut $x^{(n)}$ is itself a rational number.

DEFINITION 8.3. Let $x = a_m \dots a_0.\alpha_1 \dots$ and $y = b_k \dots b_0.\beta_1 \dots$ be two positive real numbers. Then we say that $x < y$ if, starting from the left, in the first place where the digits of x and y are not the same, the corresponding digit of x is smaller. Before making this test the infinite tails of 9's have to be converted to 0's as before. Also, if for one of the sequences the number of digits to the left of the dividing point is smaller, the missing digits are assumed to be zeros.

If both x and y have the minus sign in front of them, the order is the opposite to the one just defined. If one of them has a "+" sign and the other "−", the positive number is greater. Finally, 0 is greater than any negative number, and less than any positive number.

The next lemma corresponds to the discussion of Section 1.5.

LEMMA 8.4. *Any bounded set S of real numbers as defined above has a least upper bound.*

OUTLINE OF THE PROOF. To simplify the notation, we assume all the elements are positive. The boundedness assumption guarantees that the numbers of digits to the left of the point for any $x \in S$ is bounded by a fixed integer k. Look at the k-th digit a_k for all $x \in S$. All these digits are

0 through 9. Take the maximum of these, and call it c_k. Next, look only at the numbers $x \in S$ which have $a_k = c_k$, i.e. the maximal possible. Among those pick the maximal $(k-1)$-th digit and call it c_{k-1}. Narrow down your set further to only those numbers x whose $(k-1)$-th digit is c_{k-1} and look at the $(k-2)$-th place etc. Now it must be obvious that the real number

$$z = c_k c_{k-1} \dots c_0.\gamma_1\gamma_2 \dots$$

is the least upper bound of the set S. □

Hence, with this definition, the least upper bound property is quite easy. What is harder, however, are the definitions for $x \pm y$, xy and x/y. You certainly should know how to add finite decimal fractions, with only finitely many digits. But in doing this, you sometimes have to carry out a 1 to the left. However, when working with the infinite sequences of digits you have to work your way from the left to the right! So, suppose you have added the numbers x and y up to the n-th place, i.e. you have found $x^{(n)} + y^{(n)}$. If the $(n-1)$-th and the n-th digits to the right of the point of $x^{(n)} + y^{(n)}$ are not 9 then whatever happens to the right will not change the $(n-1)$-th digit or any of the digits to the left of it. However, if you have a tail of 9s then down the road, after looking at many, many consequent cuts, you will be still unsure if at some point you will have to carry 1 all the way to the left. And that is a problem - you don't know whether your n-th digit is going to be 9 or 0 without doing *infinitely many* computations! There are similar complications with the definitions of xy and x/y. The solution is the following:

Definition 8.5. For positive real numbers x, y the sum $x+y$ is defined as the least upper bound of the set $\{x^{(1)} + y^{(1)}, x^{(2)} + y^{(2)}, \dots\}$.

To handle subtractions, we define

Definition 8.6. $1 - 0.\alpha_1\alpha_2\dots\alpha_n \dots = 0.(9-\alpha_1)(9-\alpha_2)\dots(9-\alpha_n)\dots$

The definitions for the remaining operations are similar. The checking of the field axioms is direct but very tedious, and we will omit it.

Which of these infinite decimals correspond to rational numbers? (This was explored in some of the problems for Chapter 3.) You can easily guess the answer once you look at some examples.

Example 8.7.
(1) $11/3 = 3.6666\dots$
(2) $1/11 = 0.090909\dots$
(3) $1/7 = 0.142857142857142857\dots$

Definition 8.8. A *periodic* decimal is the one that that has a repeating pattern starting at some place.

Proposition 8.9. *Rational numbers correspond to periodic decimals.*

Proof. First take a rational number p/q and try to find its decimal expansion by repeatedly dividing with a remainder. At every step you will have a remainder. If it is zero, then we have our answer. If it is some number $n < q$ then we continue by adding a 0 to the right from it and dividing again, this time $(10n)/q$. Since there are only finitely many possibilities for the remainder – $0, 1, \dots q-1$ - at some step we will get the same remainder, and the process will repeat itself over and over again.

We illustrate this by finding the decimal expansion of $1/7$: $1 \div 7$ yields a quotient of $\mathbf{0}$ with a remainder 1. Next, $\frac{10}{7} = \mathbf{1}\,\frac{3}{7}$. Next, $\frac{30}{7} = \mathbf{4}\,\frac{2}{7}$. Next, $\frac{20}{7} = \mathbf{2}\,\frac{6}{7}$. Next, $\frac{60}{7} = \mathbf{8}\,\frac{4}{7}$. Next, $\frac{40}{7} = \mathbf{5}\,\frac{5}{7}$. Next, $\frac{50}{7} = \mathbf{7}\,\frac{1}{7}$. At this point we got $\mathbf{1}$ again, so starting from this place the pattern $\mathbf{1}428571$ is going to repeat over and over again.

To show the converse, that every decimal which starts repeating itself after some place must be rational, note that $1/9 = 0.111\dots$, $1/99 = 0.010101\dots$, $1/999 = 0.001001001\dots$ etc. Therefore, a repeating decimal

$$x = b_m \dots b_0.\beta_1 \dots b_k \alpha_1 \dots \alpha_n \alpha_1 \dots \alpha_n \dots$$

can be written as

$$b_m \dots b_0.\beta_1 \dots b_k + 10^{-k} \frac{\alpha_1 \dots \alpha_n}{10^n - 1}$$

and so is a rational number.

To illustrate this,

$$0.57123123123 \dots = 0.57 + 10^{-2} \frac{123}{999} = \frac{57}{100} + \frac{123}{99900}$$

□

2. Dedekind Cuts

This construction is motivated by the idea that each real number should be determined by the collection of rational numbers below it. Details of the following outline can be found in Rudin's text.

Definition 8.10. A *cut* is a non-empty set α of rational numbers satisfying

(1) If $x \in \alpha$, and $y < x$, then $y \in \alpha$,
(2) α has an upper bound, and
(3) α has no largest element.

The collection of all cuts is denoted by $\mathbb{R}$.

In view of (1), we could replace (2) with the assumption that α does not exhaust $\mathbb{Q}$.

We first equip $\mathbb{R}$ with an order.

Definition 8.11. Let α, β be cuts. We say $\alpha < \beta$ if and only if α is a proper subset of β.

Proposition 8.12. $(\mathbb{R}, <)$ *has the least upper bound property.*

PROOF. Let S be a non-empty collection of cuts having an upper bound β. We take γ to be the union of all the cuts comprising S. It is clear that γ satisfies the first and third conditions of Definition 8.10. Since any (rational) upper bound of β is automatically an upper bound of γ, we see that γ is in fact a cut. By definition of union, every member of S must be a subset of γ, and any set which contains each member of S must also contain γ. But that is just what it takes to make γ the least upper bound of S. □

Next, we take up addition.

DEFINITION 8.13. (1) $\alpha + \beta := \{x + y : x \in \alpha,\ y \in \beta\}$,
(2) the *zero cut* is $\theta := \{x \in \mathbb{Q} : x < 0\}$,
(3) $-\alpha := \{y \in \mathbb{Q} : -x - y \notin \alpha \text{ for some } x \in \mathbb{Q}_+\}$.

It is necessary to check that these are all cuts; the presence of x in 8.13.3 guarantees that $-\alpha$ does not have a largest member. Commutativity and associativity are clear, but it must be verified that that θ and $-\alpha$ act as additive identity and inverse respectively.

An interesting consequence of these definitions is that

$$\mathbb{R}_+ := \{\alpha \in \mathbb{R} : \alpha > \theta\} = \{\alpha \in \mathbb{R} : 0 \in \alpha\}.$$

(We have reversed the procedure of Chapter 1, where order was defined in terms of the disinguished set of positive elements.)

A somewhat subtle and long-winded definition of multiplication is needed because $\{xy : x \in \alpha \text{ and } y \in \beta\}$ is *never* a cut.

DEFINITION 8.14. Let α and β be positive cuts.
(1) $\alpha\beta = \{xy : x \in \alpha, y \in \beta, x > 0, \text{ and } y > 0\} \cup \{0\} \cup \mathbb{Q}_-$;
(2) $(-\alpha)(\beta) = (\beta)(-\alpha) = -(\alpha\beta)$,
(3) $\theta\gamma = \gamma\theta = \theta$ for every cut γ.
(4) $(-\alpha)(-\beta) = \alpha\beta$.

3. Cauchy Sequences

This construction is based on the idea that an individual real number is determined by the *collection* sequences of rationals converging to it. We use an equivalence relation to identify all such sequences.

DEFINITION 8.15. Take C to be the collection of Cauchy sequences of rational numbers. For $x, y \in C$,
(1) $x \sim y$ if and only if $\lim_{n\to\infty} x_n - y_n = 0$;
(2) $[x]$ denotes the equivalence class of x;
(3) $\mathbb{R}$ denotes the set of equivalence classes of C under $\sim$;
(4) $[x] + [y] := [x + y]$;
(5) $[x][y] := [xy]$.

It is easy to verify that $\sim$ is in fact an equivalence relation.

Two things must be checked in order for Part (4) of the definition to make sense: that sums of Cauchy sequences are again Cauchy, and that

$[x+y]$ only depends on the *classes* of x, y. The first matter is left to the reader. For the second, suppose $x \sim z$ and $y \sim w$. Then

$$\lim_{n\to\infty}(x_n + y_n) - (z_n + w_n) = \lim_{n\to\infty}(x_n - z_n) + \lim_{n\to\infty}(y_n - w_n) = 0,$$

so $[x+y] = [z+w]$. Thus addition is well defined; the argument for multiplication is the same. The field axioms also follow easily. For example, the additive identity is $[0]$, the class of the constant zero sequence.

Definition 8.16. $\mathbb{R}_+$ is the collection of equivalence classes $[x]$ such that there exists an integer N satisfying $x_n > \frac{1}{N}$ for all $n \geq N$.

Once again, it must be checked that this definition only depends on the class $[x]$, but not on the specific representative tested. It is clear that $\mathbb{R}_+$ is closed under addition and multiplication. As for trichotomy, note that if neither $[x]$ nor $-[x]$ belong to $\mathbb{R}_+$, then for each $\epsilon > 0$, we will have $|x_n| < \epsilon$ for infinitely many values of n. Since x is Cauchy, this implies $\lim_{n\to\infty} x_n = 0$, i.e., $[x] = [0]$.

It remains to check completeness.

Proposition 8.17. *As defined in this section,* $(\mathbb{R}, <)$ *has the least upper bound property.*

outline of proof. We have not explicitly mentioned our axiomitization of $\mathbb{Q}$. The simplest characterization is that $\mathbb{Q}$ is a *minimal* ordered field, that is, no proper subset of $\mathbb{Q}$ is an ordered field in its own right. It follows from this that $\mathbb{Q}$ has the archimedean property, and hence that every bounded set of integers in $\mathbb{Q}$ has a largest element.

Suppose now that S is a bounded non-empty subset of $\mathbb{R}$. For each $q \in \mathbb{Q}$, we write $\bar{q}$ for the equivalence class of the constant sequence $(q)_{n=1}^{\infty}$. Set

$$T := \{q \in \mathbb{Q} : \bar{q} \text{ is smaller than some member of } S\}.$$

For each $n \in \mathbb{N}$, we take p_n to be the largest integer such that $x_n := \frac{p_n}{10^n} \in T$. Then $x = (x_n)$ is a bounded increasing sequence. The proof is completed by checking that x is Cauchy and that $[x]$ is the least upper bound of S. $\square$

Remark 8.18. Proposition 8.17 is usually established via Proposition 2.36, but the above proof is more self-contained for our purposes.

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