Review Session

Decimal: $\sum_{n=N}^{\infty} \frac{d_n}{10^n}$ where $N \leq 1$ is an integer and each $d_n \in \{0, 1, 2, 3, \dots, 9\}$

Facts:

① Every decimal converges to a Real number
② each $x \geq 0$ has a decimal expansion
③ x will have two decimal expansions if $x = \frac{p}{10^k}$, $k \in \mathbb{Z}$ $p \in \mathbb{N}$
all other $x \geq 0$ have unique decimal exp
④ A decimal is terminating if $\exists k$ s.t. $a_i = 0 \ \forall i \geq k$
⑤ A decimal is eventually repeating if $\exists\, n, k \in \mathbb{N}$ s.t. $a_{j+k} = a_j \ \forall j \geq n$.
⑥ $x \geq 0$ has terminating decimal expansion iff $x = \frac{p}{10^k}$ $k \in \mathbb{N}$, $p \in \mathbb{N} \cup \{0\}$. (also has non terminating)
⑦ $x \geq 0$ has an eventually repeating decimal exp. iff $x \in \mathbb{Q}$
⑧ Every $x > 0$ has a non terminating expansion but 0 doesn't.

$2.357 = 2.356\overline{9}$

terminating non-terminating

Computations

① Can compute decimalexpansions of fractions and fraction for repeating decimals.

$\boxed{\text{Ex.1}}$ Express as a fraction

$x = 2.3\overline{56}$
$100x = 235.6\overline{56}$ (multiply by 100 b/c repeating block = 2)
$99x = 233.3$

$\square$

$x = \frac{233.3}{99} = \frac{2333}{990}$

geometric series

$\sum_{n=0}^{\infty} ar^n$ a = start number r = ratio

converges to $\frac{a}{1-r}$ when $|r|<1$

Prove ① Every decimal converges to a Real number.

$\boxed{\text{PF}}$ Set $a_n = \frac{d_n}{10^n}$ $b_n = \frac{9}{10^n}$ ← b/c that's biggest digit we can have

$0 \le a_n \le b_n \ \forall n$ $\sum b_n = \sum \frac{9}{10^n}$ geometric series with $r = \frac{1}{10}$ b_n conv. b/c $|r|<1$

By comparison test, a_n converges.

Review Session

Series with sign changes

Two Theorems

① If $\sum |a_n|$ converges, then $\sum a_n$ converges.

② (ALT Series test) if $b_n \downarrow 0$, then $\sum (-1)^n b_n$ converges

$\sum a_n$	$\sum \lvert a_n \rvert$	$\sum a_n$
conv	conv	conv abs
conv	DIV	conv. conditionally
DIV	DIV	DIV

Strategy to classify $\sum a_n$

① Examine $\sum |a_n|$, if conv, done
if $|a_n| \not\to 0$, then $a_n \not\to 0$ then $\sum a_n$ DIV (nth term)
- otherwise we know $\sum a_n$ either DIV or conv conditionally

② if ALT series applies, $\sum a_n$ conv conditionally
Otherwise, CRY

$\sum a_n$	$\sum b_n$	$\sum (a_n + b_n)$
conv. abs	conv abs	conv. abs
conv abs	DIV	DIV
conv. cond	DIV	conv cond. or abs (if $a_n = -b_n$)
conv. cond	conv abs	conv. cond

PF: $\sum a_n$, $\sum b_n$ conv $\Rightarrow \sum a_n + b_n$ conv.
Suppose, for nefarious purpose (contrary), $\sum |a_n + b_n|$ conv.
we also know $\sum -b_n$ conv. ABS.
So $\sum (a_n + b_n) + (-b_n) = \sum a_n$ should mean that
$\sum a_n$ converges absolutely. $\Rightarrow\Leftarrow$.
$\therefore \sum a_n + b_n$ conv. conditionally. $\square$

① * $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(\ln(\ln n))}$

$\ln n < n$

$\ln(\ln n) < \ln n < n$

$\ln(\ln(\ln n))) < \ln(\ln n) < n$

$\frac{1}{\ln(\ln(\ln n)))} > \frac{1}{n}$ so DIV by comparison

but * conv. by alt. series test, so conv conditionally

② $\sum \frac{(-1)^n n^2}{3n^2+5n+7}$ DIV by n'th term test

③ $\sum \frac{1}{\cos(n)}$ DIV

Review Session: chp 4

* **Most Important**: Defn of Limit, continuous

Defn: f is continuous @ a if given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. $|f(x) - f(a)| < \varepsilon$ whenever $|x-a| < \delta$

Defn: $\lim_{x \to a} f(x) = L$ Given $\varepsilon > 0$, $\exists\, \delta > 0$ s.t. when $0 < |x-a| < \delta$, $|f(x) - L| < \varepsilon$

Computations

(1) $\lim_{x \to 3} 5x + 2 = 17$

(2) $\lim_{x \to 3} x^2 = 9$ $\quad f(x) = x^2 \quad L = 9 \quad a = 3$

Scratch

$|x^2 - 9| < \varepsilon \quad |x^2-9| = |(x+3)(x-3)| \le |x+3||x-3|$ we know we can control $x-3$ so

$\rightarrow -1 < x-3 < 1 \rightarrow 2 < x < 4 \rightarrow 5 < x+3 < 7$

$|x-3| < 1$ so $|x+3| < 7 \quad |x^2-9| \le 7|x-3| < \varepsilon \quad |x-3| < \varepsilon/7$

Given $\varepsilon > 0$. Take $\delta = \min(1, \varepsilon/7)$.
Assume $|x-3| < \delta$. Then $|x-3| < 1$ and $|x-3| < \varepsilon/7$.
Since $|x-3| < 1$ we know $|x+3| < 7$. So
$|x^2-9| = |x+3||x-3| < 7|x-3| < 7(\varepsilon/7) = \varepsilon$

OR

$|x+3| = |(x-3) + 6| \le |x-3| + 6 < 1 + 6 = 7$

Computing the limit using definition

① $\lim_{x\to a} f(x) = L \qquad \lim_{x\to a} g(x) = M$

then $\lim_{x\to a} f(x) + g(x) = L + M$

Theorems for chp. 4

① Intermediate Value Theorem: solve ineq. or show ~~f~~ has solutions
② Max Value → closed interval
③ Rolle's Theorem
④ Mean Value
⑤ Generalized Mean Value Theorem

Prove $x^5 + x^3 = 79$ has a solution that is unique
$f(x) = x^5 + x^3 - 79$ cont.
$f(0) = -79$ $\therefore$ by intermediate value theorem we know
$f(5) = +$ it has a solution

- the solution is unique b/c $f'(x) = 5x^4 + 3x^2 > 0$
- so it will ALWAYS be positive and by Rolle's Theorem
it has to be unique.

Review Session: Chp 5

Definitions

① Taylor or MacLaurin Polynomials
② " " Series
③ n'th remainder $r_n(x) = f(x) - p_n(x)$

Theorems

Taylor's - start with f @ pt. a.
① @ a: $p_n^{(k)}(a) = f^{(k)}(a)$ for $k = 0, \ldots, n$
② near a: $\lim_{x \to a} \frac{f(x) - p_n(x)}{(x-a)^n} = 0$
③ Error quantification: $r_n(x) = \frac{f^{(n+1)}(c)(x-a)^{n+1}}{(n+1)!}$ where c btwn a + x

Computations

chasing

Approximate sin(1)
a) Use 5^{th} deg mac polynomial to approx. sin(1).
$f(x) = \sin x$ $a = 0$ $x = 1$

Mac Series for $\sin x \sim \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} + \ldots$

$p_5(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$

$\sin(1) = f(1) \sim p_5(1) = 1 - \frac{1}{6} + \frac{1}{120}$

b) Give an upper bound for error in (a)

$r_5(x) = \frac{f^6(c)\, x^6}{6!}$ $|r_5(1)| = \left| \frac{\sin c\,(1)}{6!} \right| < \frac{1}{6!} = \frac{1}{720} < .01 < 10^{-2}$

c) Efficient method to find estimate of $\sin(1)$ with error $< 10^{-5}$

$$r_n(x) = \frac{f^{(n+1)}(c)\,(x-a)^{n+1}}{(n+1)!}$$

$$r_n(1) = \frac{f^{(n+1)}(c)\,(1-a)^{n+1}}{(n+1)!}$$ since $a=0$ then $(1)^{n+1}=1$

$$r_n(1) = \frac{\sin/\cos(c)}{(n+1)!} \leq \frac{1}{(n+1)!} \underset{<}{\text{WANT}} \frac{1}{10^5}$$

Hey!

Hey! $(n+1)! > 10^5$

Hey! $9! > 10^5$ $\boxed{n=8}$

① $\sum_{n=0}^{\infty} 3^n (x-4)^n$ We know

Review Session: Chp. 5

① $\sum_{n=0}^{\infty} 3^n (x-4)^n$ we know $\sum_{n=0}^{\infty} y^n \sim \frac{1}{1-y}$

$= \sum_{n=0}^{\infty} (3(x-4))^n \sim \frac{1}{1-(3(x-4))}$ • substitution

② $\sum_{n=0}^{\infty} \frac{x^n}{2^n} = \sum_{n=0}^{\infty} \left(\frac{x}{2}\right)^n = \frac{1}{1-\frac{x}{2}} = \frac{2}{2-x}$ • substitution

③ $\sum_{n=1}^{\infty} n x^{n-1}$ • this is derivative of x^n so take derivative of $\frac{1}{1-x}$

$\sum_{n=1}^{\infty} n x^{n-1} \sim \frac{1}{(1-x)^2}$

④ $\sum_{n=1}^{\infty} n x^n \sim \frac{x}{(1-x)^2}$ • multiply ③ by x

⑤ $\sum_{n=1}^{\infty} n^2 x^n$ • when we take derivative of nx^n we get $n^2 x^{n-1}$

$f(x) = \frac{x}{(1-x)^2}$ $f'(x) = \frac{(1-x)^2(1) + (x)(2)(1-x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$

• Now we multiply by x

$\sum_{n=1}^{\infty} n^2 x^n \sim \frac{x+x^2}{(1-x)^3}$

⑥ $\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$ • integrate

$$\int x^n = \frac{x^{n+1}}{n+1} = \frac{x^n}{n}$$

$$\int \frac{1}{1-x}\,dx = -\ln(1-x)$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} \approx -\ln(1-x)$$

⑦ $\sum \frac{x^n}{n^2} = \frac{x}{1} + \frac{x^2}{4} + \frac{x^3}{9} + \dots$ ① divide $\frac{x^n}{n}$ by x

$\dfrac{\sum \frac{x^n}{n}}{x} = \sum \frac{x^{n-1}}{n} = 1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^2}{3} \dots$ • now we can integrate

$$= -\frac{\ln(1-x)}{x}$$

$$\int -\frac{\ln(1-x)}{x}$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2} = \int_0^x -\frac{\ln(1-t)}{t}\,dt$$

$$\sum \frac{t^{n-1}}{n} \sim -\frac{\ln(1-t)}{t}$$

$$\int_0^x \sum \frac{t^{n-1}}{n} \sim \int_0^x -\frac{\ln(1-t)}{t}\,dt$$

$$\Downarrow$$

$$\sum \int_0^x \frac{t^{n-1}}{n}\,dt \qquad \sum_{n=1}^{\infty} \frac{t^n}{n^2}\Big|_0^x = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$