Notes to Final Exam (200 points; median 149)

Include full justification for your work. You may appeal to statements proved in the text, but not to results of homework problems.

1 (20pts; median: 85%). Conformal maps

a) Find, with proof, all Mobius transformations mapping $\mathbb{D}$ onto itself.

b) Give an example of a region which is conformally equivalent to $\mathbb{D}$, but for which the equivalence cannot be implemented by a Mobius transformation.

Solution. For Part a), you should first answer the question, then briefly address sufficiency, and then establish necessity.

The answer to Part a) is $Tz = c\frac{z-a}{1-\overline{a}z}$ where $|c| = 1$ and $|a| < 1$.

For sufficiiency, take T as above. Then for $|z| = 1$, we have

$$|Tz| = \left|\frac{z-a}{\overline{z}(1-\overline{a}z)}\right| = \left|\frac{z-a}{\overline{z}-\overline{a}}\right| = 1,$$

so T maps the unit circle to itself. Since $|T(0)| = |a| < 1$, we see by connectedness that $T(\mathbb{D}) \subset \mathbb{D}$.

For necessiity, suppose T is a Mobius transformation mapping $\mathbb{D}$ into itself. By continuity, T must map the unit circle to itself. Choose a with $T(a) = 0$ whence $T(\frac{1}{\overline{a}}) = \infty$ by symmetry. For $a \neq 0$, this yields

$$Tz = \lambda\frac{z-a}{z-\frac{1}{\overline{a}}} = -\lambda\overline{a}\frac{z-a}{1-\overline{a}z},$$

for some $\lambda \in \mathbb{C}$. Take $c := -\lambda\overline{a}$. From $|T1| = 1$, we then conclude that $|c| = 1$.

It is also possible to apply the text's discussion of the maps ϕ_a following the Schwarz lemma by first recalling that all Mobius transformations are bijective.

For Part b), you can take G to be a half disc. It is conformally equivalent to $\mathbb{D}$ by the Riemann mapping theorem (or by direct construction); however Mobius trnasformations must take circles to lines or circles and hence can only take $\mathbb{D}$ to disks or half-planes.

2 (20pts; median: 65%). Find all entire functions f which satisfy $(f(x))^2 = \exp x$ for all $x \in \mathbb{R}$.

Solution. The two solutions are $f(z) = \exp(\frac{z}{2})$ and $f(z) = -\exp(\frac{z}{2})$. It is clear that these work. To see that they are the only possibilities, note that $A := \{z \in \mathbb{C} : f(z) = \exp(\frac{z}{2})\}$ and $B := \{z \in \mathbb{C} : f(z) = -\exp(\frac{z}{2})\}$ are disjoint closed sets whose union is all of $\mathbb{C}$, whence the result follows by connectedness.

It is also possible to match the Taylor series of $(f(z))^2$ with that of $\exp(z)$.

3 (25pts; median 40%). Suppose f has a non-removable isolated singularity at $z = a$. Prove that $\exp \circ f$ has an essential singularity at a.

Proof. Assuming f has an essential singularity at a, the Casorati-Weierstrauss Theorem provides sequences $(z_n), (w_n)$, both of which converge to a with $\lim f(z_n) = 0$ and $\lim f(w_n) = \pi i$. Thus $\lim \exp(f(z_n)) = 1$, while $\lim(\exp f(w_n)) = -1$. Thus $\lim_{z\to a} \exp(f(z))$ fails to exist in $\mathbb{C}_\infty$, and $\exp \circ f$ has an essential singularity at a.

On the other hand, if f has a pole at a, then $\frac{1}{f}$has a zero at a. Fix $n \in \mathbb{N}$. By the open mapping theorem, $f(Ball(0, \frac{1}{n}))$ covers a deleted neighborhood of ∞ and thus there exist z_n, w_n in $Ball(a, \frac{1}{n})$ whose images under f are odd and even integral multiples of πi respectively. Thus $\exp(f(z_n)) = -1$ while $\exp(f(w_n)) = 1$. Since both sequences $(z_n), (w_n)$ converge to a, we again conclude that $\lim_{z\to a} \exp(f(z))$ does not exist in $\mathbb{C}_\infty$, and hence that $\exp \circ f$ has an essential singularity at a.

There are other arrangements of the proof. One may not however "take logarithms". Indeed, while f isself is branch of $\log \exp f$, the range of $\exp f$ always includes circles of large radiii centered at 0 and thus there is no branch of log defined on the range of $\exp f$.

Another approach which doesn't work is substituting the Laurent series of f into the Taylor series of exp. While the rearrangement of the double series needed to express it as a single Laurent series can be justified, there is no easy way to guarantee that any of the resulting coefficients of negative powers of $z - a$ fail to vanish.

4 (25pts; median 100%). Suppose f is entire and has a pole at ∞. Prove that the range of f is all of $\mathbb{C}$.

Proof. This problem was on the midterm. The easiest approach is to note that if the range of f omitted some value c, then g defined by $g(z) = \frac{1}{f(z)-c}$ would be bounded and entire, hence constant by Liouville's Theorem, a contradiction.

Another approach is to note that the function h given by $h(z) = f(\frac{1}{z})$ has a pole at 0 and is bounded at ∞. That means the Laurent series of h does not involve any strictlly positive powers of z, whence f is a non-constant polynomial.

5 (25pts; median 68%). Suppose $n \geq 2$. Use a wedge of angle $\frac{2\pi}{n}$ to evaluate $I := \int_0^\infty \frac{dx}{1+x^n}$.

Solution. Set $f(z) = \frac{1}{1+z^n}$ and write Γ_R for the suggested contour. For $R > 1$, f has a single pole inside this wedge, namely $p := \exp(\frac{\pi i}{n})$. By L'hopital's rule, the residue at this pole is $\lim_{z\to p} \frac{z-p}{z^n+1} = \frac{1}{np^{n-1}} = \frac{-p}{n}$. The residue theorem thus yields

$$(2\pi i)\left(\frac{-p}{n}\right) = \int_0^R f(x)\,dx + \int_{arc_R} f(z)\,dz - \int_{[0,p^2R]} f(z)\,dz.$$

The middle integral approaches zero as $R \to \infty$, but the last integral does not; in fact, substitutiing $z = tp^2$,with $0 \leq t \leq R$ yields $\int_{[0,p^2R]} f(z)\,dz = p^2 I$. Taking the limit in the display as $R \to \infty$ thus yields $\frac{-2\pi ip}{n} = (1-p^2)I$. Dividing by $-2ip$ leads to $\frac{\pi}{n} = \frac{p^2-1}{2ip}(I) = \left(\sin\frac{\pi}{n}\right) I$ whence $I = [\frac{\pi}{n}]/[\sin\frac{\pi}{n}]$.

6 (20pts; median 95%). If $a > e$, prove that the equation $az^n = e^z$ has n roots in $\mathbb{D}$.

Proof. Apply Rouche's Theorem on the unit circle with $f(z) = az^n$, and $g(z) = e^z - az^n$.

7 (20pts; median 73%). Suppose $f : \mathbb{D} \to \mathbb{D}$ is analytic and has a zero of order $n \in \mathbb{N}$ at 0. Prove that $|f(z)| \le |z^n|$ for all $z \in \mathbb{D}$ and obtain a sharp upper bound on $|f^{(n)}(0)|$.

Solution. When $n = 1$, this is Schwarz's Lemma. To handle the general case, take $g(z) := \begin{cases} \frac{f(z)}{z^n}, & z \ne 0 \\ \frac{f^{(n)}(0)}{n!}, & z = 0 \end{cases}$. The Maclaurin expansion of f shows that g is analytic in $\mathbb{D}$. For $|z| = r < 1$, we have $|g(z)| \le \frac{1}{r^n}$. Applying the maximum modulus theorem and allowing $r \to 1^-$, we see that $|g(z)| \le 1$ for all $z \in \mathbb{D}$. This shows $|f(z)| \le |z^n|$ throughout $\mathbb{D}$ and provides the upper bound $|f^{(n)}(0)| \le n!$. The function $f(z) = z^n$ shows this upper bound is sharp.

It is also possible to give an inductive proof of this result.

8 (20pts; median 100%). Let $\mathcal{F}$ be a normal family in $H(G)$ and take $\mathcal{A}$ to be the set of all averages of its members, i.e., $\mathcal{A} := \{\frac{f_1+\cdots+f_n}{n} :$ each $f_j \in \mathcal{F}\}$. Prove that $\mathcal{A}$ is also normal.

Proof. This is an application of Montel's Theorem.

9 (25pts; median: 44%). Let G be a region which contains the closed disk of radius r about a point a. Suppose u is a harmonic function on G which vanishes on the circle $|z-a| = r$. Prove that u vanishes on all of G.

Proof. The maximum and minimum principles show that u vanishes on $Ball(a,r)$. The proof can be completed via a connectedness argument as in the last homeowrk assignment.

Unless G is simply connected, it is not true that every harmonic function has a harmonic conjugate on G. Ms. Ghosh Hajra thought of a neat way around this. Define $f = u_x - iu_y$. The Cauchy-Riemann equations show f is analytic on G, and must vanish on $Ball(a,r)$ since u does. That makes $f \equiv 0$ on G which forces u to be constant there. (The construction of f makes sense for any harmonic u and region G, but f may fail to have an antiderivative throughout G when that domain is not simply connected.)